Diakoptic and large change sensitivity analysis

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Indexing terms: Networks, Sensitivity

Abstract: An approach to the analysis of large circuits based on the use of the large change sensitivity technique applied to decomposed networks is presented. As a result of this approach a simple, compact notation for the solution vector is derived. The method is applicable to nonlinear analogue networks with hierarchical decomposition simulated by inserted ideal switches. A simple illustrative example is given.

1 Introduction

Many techniques have been developed to analyse partitioned networks to reduce computational effort and to save computer storage space. These techniques originated with diakoptics which was introduced by Kron [1]. The partitioning and sparse matrix techniques were later combined, producing yet more efficient methods [2–4]. Matrix modification techniques were used [5–7] to simplify analysis in cases when only some coefficients change in the system equations. Further development included application of the hierarchical decomposition approach to network analysis, which allows analysis of very large networks with great efficiency [8, 9]. The reader may refer to References 10 and 11 for a discussion of the efficiency of these approaches.

A common problem of the decomposition techniques is the complexity of the algorithms and the high level of abstraction used. Techniques are made available for readers who are not experts in computer aided analysis. A compact notation is derived for the analysis of a partitioned network based on the large change sensitivity approach [5]. The large change sensitivity approach allows the solution of the partitioned network, simply explaining the influence of the subnetworks on the solution of the undivided network. The proposed method is applicable to nonlinear analogue networks, although it is illustrated with a linear example to maintain simplicity.

This paper, like recent work by Rohrer [12], is yet another attempt to simplify the analysis of decomposed networks.

2 Equations of a decomposed nonlinear network

Consider a large nonlinear network decomposed into s subnetworks. The separation of subnetworks can easily be achieved by inserting ideal switches between the interconnection nodes (Fig. 1). When the switches are open the network is decomposed and each subnetwork can be solved separately. When the switches are closed the original network is obtained. The objective is to update solutions in each subnetwork according to the rules of the large change sensitivity approach [5]. For simplicity of presentation it is assumed that each subnetwork contains a common reference node. This requirement does not restrict the application of the proposed approach. A general paritition can be implemented as discussed in Reference 8.

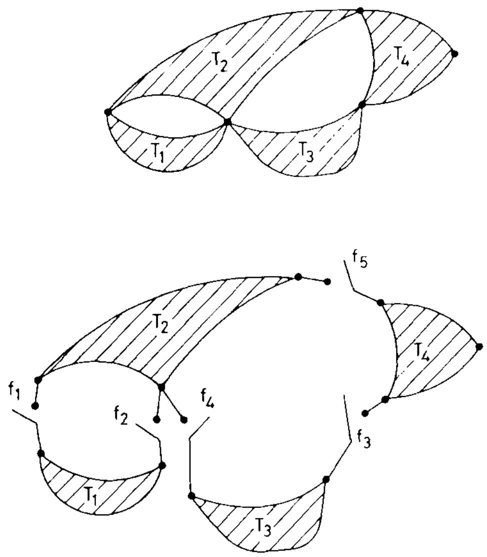

Fig. 1 *Network T decomposed by ideal switches*

Each subnetwork can be described by a, possibly nonlinear, vector equation

$$g_i(y_i) = \mathbf{0} \quad i = 1, 2, \ldots, s \tag{1}$$

where the independent variables y_i represent either nodal voltages or branch currents. Both vectors g_i and y_i have the same dimension n_i.

If two nodes j and m are connected by an ideal switch f (Fig. 2), an unknown current i_f is added to the Kirchhoff

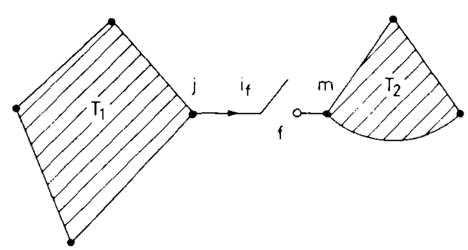

Fig. 2 *Effect of ideal switch*

Paper 8367G (E10), first received 3rd January and in revised form 8th July 1991

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current law (KCL) equation at the node j and i_f is subtracted from the KCL equation at the node m. Eqn. 1 is then augmented by an additional equation

$$(v_j - v_m)F + (F - 1)i_f = 0 \qquad (2)$$

in which v_j is an element of the vector y_1, v_m is an element of y_2 and i_f is an additional variable. The value F is 0 for the open switch and 1 for the closed switch [5]. Overall, add t such equations where t is the number of switches used for interconnections.

The system of nonlinear equations for the interconnect network $g(y) = \mathbf{0}$ is solved through the Newton–Raphson iterative process based on

$$\frac{\partial g(y^k)}{\partial y} \Delta y^k = -g(y^k) \qquad (3)$$

where Δy^k is a $n \times 1$ vector of the incremental changes in the kth iteration and n is the number of unknown currents and voltages plus the number of switches. The Jacobian of the system equations has the following form:

$$\frac{\partial g(y^k)}{\partial y} = \left[\begin{array}{cccc|c} \frac{\partial g_1}{\partial y_1} & & & & \lambda_1 \\ & \frac{\partial g_2}{\partial y_2} & & & \lambda_2 \\ & & \ddots & & \vdots \\ & & & \frac{\partial g_s}{\partial y_s} & \lambda_s \\ \hline F_1 & F_2 & \cdots & F_s & (F-1)\mathbf{I} \end{array}\right] \qquad (4)$$

where

$$y^k = \begin{bmatrix} y_1^k \\ y_2^k \\ \vdots \\ y_s^k \\ i_f^k \end{bmatrix}$$

$$g(y^k) = \begin{bmatrix} g_1(y_1^k) + \lambda_1 i_f^k \\ g_2(y_2^k) + \lambda_2 i_f^k \\ \vdots \\ g_s(y_s^k) + \lambda_s i_f^k \\ \sum_{i=1}^{s} F\lambda_i^T y_i^k + (F-1)\mathbf{I} i_f^k \end{bmatrix}, \quad i_f^k = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_t \end{bmatrix} \qquad (5)$$

and

$$F_i = F \cdot \lambda_i^T \qquad (6)$$

The element l_{jf} of the incidence matrix λ_i is as follows:

$$l_{jf} = \begin{cases} 1 & \text{if } i_f \text{ is directed out of the node j} \\ -1 & \text{if } i_f \text{ is directed toward the node j} \\ 0 & \text{if } i_f \text{ is not incident on the node j} \end{cases}$$

At this part of the description all switches are assumed to operate simultaneously; thus, F is a scalar representing the state of all switches. The advantages of operating the switches in some sequence will also be considered.

3 System solution

The linear system eqn. 3 can be efficiently solved using the large change sensitivity approach [5]. If the solution to the linear system eqn. 3 is known for 'nominal' values of circuit parameters, then the large change sensitivity approach provides a method of solving eqns. 3 for a new values of parameters by updating the nominal solution. Only the changes in ideal switches are considered and extension to a case of large changes in other parameter characteristics is straightforward. By the 'nominal' case the eqn. 3 is considered with variable F set to zero, i.e. all the switches open. In such a case eqn. 3 can be replaced by

$$\mathbf{T}_0 \mathbf{X}_0 = \mathbf{W}_0 \qquad (7)$$

where

$$\mathbf{T}_0 = \left.\frac{\partial g(y^k)}{\partial y}\right|_{F=0} = \left[\begin{array}{cccc|c} \frac{\partial g_1}{\partial y_1} & & & & \lambda_1 \\ & \frac{\partial g_2}{\partial y_2} & & & \lambda_2 \\ & & \ddots & & \vdots \\ & & & \frac{\partial g_s}{\partial y_s} & \lambda_s \\ \hline & & & & -\mathbf{I} \end{array}\right] \qquad (8)$$

and

$$\mathbf{X}_0 = \Delta y^k \Big|_{F=0} \quad \mathbf{W}_0 = -g(y^k)\Big|_{F=0} \qquad (9)$$

Because of the structure of $\mathbf{T}_0$, eqn. 7 can be solved without difficulties as the inverse of $\mathbf{T}_0$ is (the reader may wish to verify that the product $\mathbf{T}_0 \mathbf{T}_0^{-1} = \mathbf{I}$)

$$\mathbf{T}_0^{-1} = \left[\begin{array}{cccc|c} \left(\frac{\partial g_1}{\partial y_1}\right)^{-1} & & & & \left(\frac{\partial g_1}{\partial y_1}\right)^{-1}\lambda_1 \\ & \left(\frac{\partial g_2}{\partial y_2}\right)^{-1} & & & \left(\frac{\partial g_2}{\partial y_2}\right)^{-1}\lambda_2 \\ & & \ddots & & \vdots \quad \vdots \\ & & & \left(\frac{\partial g_s}{\partial y_s}\right)^{-1} & \left(\frac{\partial g_s}{\partial y_s}\right)^{-1}\lambda_s \\ \hline & & & & -\mathbf{I} \end{array}\right] \qquad (10)$$

In fact, each subnetwork can be analysed separately; thus, if multiple processors are available, the computation can be done more quickly by computing the elements of $\Delta y^k|_{F=0}$ in parallel. When all switches are closed, corresponding to setting variable F to unity, the interconnected network is obtained and eqn. 3 can be replaced by

$$\mathbf{T}\mathbf{X} = \mathbf{W} \qquad (11)$$

where, from eqns. 4 and 5

$$\mathbf{T} = \left.\frac{\partial g(y^k)}{\partial y}\right|_{F=1} = \left[\begin{array}{cccc|c} \frac{\partial g_1}{\partial y_1} & & & & \lambda_1 \\ & \frac{\partial g_2}{\partial y_2} & & & \lambda_2 \\ & & \ddots & & \vdots \\ & & & \frac{\partial g_s}{\partial y_s} & \lambda_s \\ \hline \lambda_1^T & \lambda_2^T & & \lambda_s^T & \end{array}\right] \qquad (12)$$

and

$$X = \Delta y^k \Big|_{F=1}$$

$$W = -g(y^k) \Big|_{F=1} \tag{13}$$

From the large change sensitivity approach, it follows that the solution vector X can be obtained by updating X_0 as:

$$X = X_0 - T_0^{-1} P z \tag{14}$$

where z is a $t \times 1$ vector obtained from

$$(I + Q^T T_0^{-1} P) z = Q^T X_0 \tag{15}$$

P and Q are the $n \times t$ topological matrices which indicate the location of the switches

$$P = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \quad Q = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_s \\ I \end{bmatrix} \tag{16}$$

Using eqns. 10 and 16, eqn. 15 can be simplified to

$$\left[\sum_{i=1}^{s} \lambda_i^T \left(\frac{\partial g_i}{\partial y_i} \right)^{-1} \lambda_i \right] z = Q^T X_0 \tag{17}$$

Each λ_i has, at most, one nonzero value (1 or -1) in each column and only f_i columns are nonzero, where f_i is the number of switches incident on the ith subnetwork. Therefore, each product $\lambda_i^T (\partial g_i / \partial y_i)^{-1} \lambda_i$ contains elements of $(\partial g_i / \partial y_i)^{-1}$ obtained on the intersection of the rows and columns which correspond to the nodes incident on the switches. From eqn. 14

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_s \\ X_{s+1} \end{bmatrix} = \begin{bmatrix} X_{10} \\ X_{20} \\ \vdots \\ X_{s0} \\ 0 \end{bmatrix} - \begin{bmatrix} \left(\frac{\partial g_1}{\partial y_1} \right)^{-1} \lambda_1 \\ \left(\frac{\partial g_2}{\partial y_2} \right)^{-1} \lambda_2 \\ \vdots \\ \left(\frac{\partial g_s}{\partial y_s} \right)^{-1} \lambda_s \\ -I \end{bmatrix} z \tag{18}$$

so the subvectors of the solution vector X can be individually corrected and a parallel technique can be used to reduce the processing time. Note that since a subvector $X_i = \Delta y_i^k |_{F=1}$, its dimension is the same as the number of variables in a subnetwork T_i, i.e. n_i. The last subvector X_{s+1}, which represents increments in switch currents, has t components.

The subvectors X_{i0} can be obtained as

$$X_{i0} = \left(\frac{\partial g_i}{\partial y_i} \right)^{-1} [-g_i(y_i^k) - \lambda_i i_f^k] \tag{19}$$

therefore each subvector X_i can be evaluated independently using

$$X_i = \left(\frac{\partial g_i}{\partial y_i} \right)^{-1} [-g_i(y_i^k) - \lambda_i (i_f^k - z)] \tag{20}$$

It is obvious that the subvectors X_i can be obtained without inversion of the coefficient matrices $\partial g_i / \partial y_i$; their LU factorisations (or equivalent) will be applied to the right hand side vectors $[-g_i(y_i^k) - \lambda_i (i_f^k - z)]$.

4 Hierarchical decomposition

The partition and large change sensitivity approach can also be used in the case of hierarchical decomposition [8, 9, 13]. The concept of hierarchical decomposition can be explained by analysing eqn. 17. The coefficient matrix on the left hand side of this equation may be big enough to justify its decomposition. In the proposed switch based approach, this means that not all switches will be closed at the same time. As an effect of closing the first level switches, small subnetworks will first be combined into larger subnetworks and the solution vector X_0 will be updated to X_1. The next group of switches will then be closed to obtain larger subnetworks and the solution vector X_1 updated to X_2. This procedure is repeated until the entire network is put back together.

As an example consider the decomposed network from Fig. 1. If all the switches $f_1, \ldots, f_5$ are open, the network is decomposed into four subnetworks $T_1, \ldots, T_4$. At the first step, close the switches f_1, f_2 and f_3 to obtain two subnetworks; T_{12} which combines T_1 and T_2, and T_{34} which combines T_3 and T_4. Then the switches f_4 and f_5 are closed to obtain the entire network T. This hierarchical decomposition is illustrated in Fig. 3. Decomposition

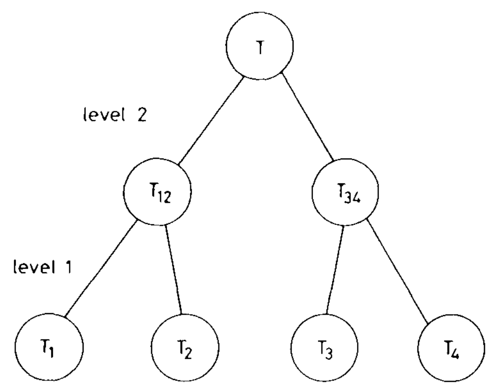

Fig. 3 *Hierarchical decomposition*

levels shown in Fig. 3 reveal which subnetworks are combined to obtain a bigger subnetwork at each step of this process.

If the switches are numbered monotonically in the sequence in which they are closed, then the general organisation of the Jacobian matrix, in a case of hierarchical decomposition, is as follows:

$$\frac{\partial g}{\partial y} = \left[\begin{array}{cc|cc|c|c} \frac{\partial g_1}{\partial y_1} & & & & \Lambda_1 & \\ & \frac{\partial g_2}{\partial y_2} & & & & \Lambda_2 \\ \hline & & \frac{\partial g_3}{\partial y_3} & & \Lambda_1 & \\ & & & \frac{\partial g_4}{\partial y_4} & & \\ \hline F_1 & & F_1 & & (F-1)I & \\ & F_2 & & & & \end{array} \right] \tag{21}$$

where F_k and Λ_k represent submatrices of F_i and λ_i which correspond to the switches closed at level k of the hierarchical decomposition. The solution procedure in the hierarchical partition approach is organised as discussed earlier. Subnetworks from a given decomposition level are combined to obtain a solution of subnetworks on a higher level. These combination operations can be performed in parallel since the various subnetworks on the higher level do not interact.

5 Example

A simple example with only one level of decomposition is used to illustrate the proposed technique. Consider the linear active network shown in Fig. 4. Impedances of all

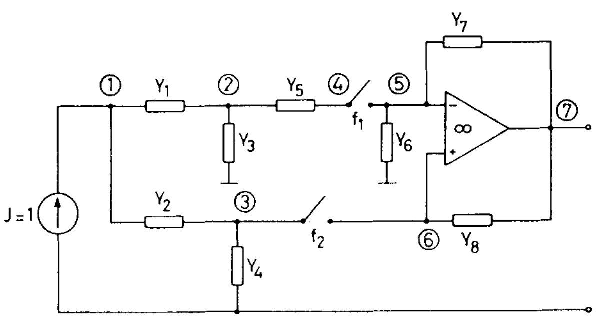

Fig. 4 *Example network*

network elements are assumed to be 1 Ω. The network is decomposed into two subnetworks T_1 and T_2 by opening the switches f_1 and f_2. The vector eqn. 1 for such a network can be written, using the modified nodal approach [5], as

$$g(y) = Ty - J = 0 \tag{22}$$

From eqn. 22, the system of eqn. 11 has the form

$$TX = \begin{array}{c|cccc|cccc|c|c|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & & & & & & & & & & \\ 2 & T_1 & & & & & & & & & \\ 3 & & & & & & & & & & 1 \\ 4 & & & & & & & & & 1 & \\ \hline 5 & & & & & & & & & -1 & \\ 6 & & & & & & & T_2 & & & -1 \\ 7 & & & & & & & & & & \\ 8 & & & & & & & & & & \\ \hline 9 & & & & F & -F & & & & F-1 & \\ 10 & & & F & & & -F & & & & F-1 \end{array}$$

$$\times \begin{bmatrix} \Delta v_1 \\ \Delta v_2 \\ \Delta v_3 \\ \Delta v_4 \\ \Delta v_5 \\ \Delta v_6 \\ \Delta v_7 \\ \Delta i_0 \\ \Delta i_1 \\ \Delta i_2 \end{bmatrix} = W \tag{23}$$

where

$$W = -Ty^k + J$$

$$X = \Delta y^k \Big|_{F=1} \tag{24}$$

and

$$(y^k)^T = [v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \quad i_0 \quad i_1 \quad i_2] \tag{25}$$

$v_1, \ldots, v_7$ are the node voltages, i_0 is the output current from the operational amplifier, i_1 and i_2 are the switch currents and $J^T = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$ is the vector of current excitations. Note that, if the initial guess for the system variables is $y^0 = 0$, then eqn. 23 reduces to

$$TX = J \tag{26}$$

Thus, as one would expect, for a linear network only one iteration of the system of equations is necessary. However, such a simple reduction does not occur for nonlinear networks. Matrices T_1 and T_2 are obtained using the modified nodal approach [5] as

$$T_1 = \begin{bmatrix} Y_1 + Y_2 & -Y_1 & -Y_2 & 0 \\ -Y_1 & Y_1 + Y_3 + Y_5 & 0 & -Y_5 \\ -Y_2 & 0 & Y_2 + Y_4 & 0 \\ 0 & -Y_5 & 0 & Y_5 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} Y_6 + Y_7 & 0 & -Y_7 & 0 \\ 0 & Y_8 & -Y_8 & 0 \\ -Y_7 & -Y_8 & Y_8 + Y_7 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

First the nominal system (with $F = 0$) is solved to obtain $\hat{X}_0$.

$$\hat{X}_0^T = [\hat{X}_{10}^T \quad \hat{X}_{20}^T \quad 0 \quad 0]$$

where

$$\hat{X}_{10} = \begin{bmatrix} \Delta v_{10} \\ \Delta v_{20} \\ \Delta v_{30} \\ \Delta v_{40} \end{bmatrix} = T_1^{-1} W_1 = \frac{1}{4} \begin{bmatrix} 4 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 \\ 2 & 1 & 3 & 1 \\ 2 & 3 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\hat{X}_{20} = \begin{bmatrix} \Delta v_{50} \\ \Delta v_{60} \\ \Delta v_{70} \\ \Delta i_{00} \end{bmatrix} = T_2^{-1} W_2 = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & -2 \\ 1 & -2 & 0 & -2 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Incidence matrices λ_1 and λ_2 are as follows:

$$\lambda_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\lambda_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Next, the correction vector z is calculated from eqn. 17 as follows:

$$(\lambda_1^T T_1^{-1} \lambda_1 + \lambda_2^T T_2^{-1} \lambda_2) z = Q^T \hat{X}_0$$

or

$$\left\{ \frac{1}{4} \begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right\} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \Delta v_{40} & -\Delta v_{50} \\ \Delta v_{30} & -\Delta v_{60} \end{bmatrix}$$

which yields

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The solution vector can be updated according to eqn. 18 as follows:

$$\Delta y_1^0 = X_1$$

$$= X_{10} - T_1^{-1} \lambda_1 z$$

$$= \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & 2 \\ 3 & 1 \\ 1 & 3 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ -2 \\ -2 \end{bmatrix}$$

$$\Delta y_2^0 = X_2$$

$$= \hat{X}_{20} - T_2^{-1} \lambda_2 z$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -2 \\ -5 \\ 6 \end{bmatrix}$$

and

$$\Delta y_3^0 = X_3 = z = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

For linear systems the obtained X is the final solution. Combining the results obtained, the following solution vector is therefore obtained:

$$(y^1)^T = X^T$$

$$= [v_1 \ v_2 \ v_3 \ v_4 \ v_5 \ v_6 \ v_7 \ i_0 \ i_1 \ i_2]$$

$$= [-1 \ -1 \ -2 \ -2 \ -2 \ -2 \ -5 \ 6 \ 1 \ 3]$$

The same result is obtained in the direct analysis.

6 Conclusion

The large change sensitivity approach can be used to analyse a decomposed network. Through the use of ideal switches, the process of decomposition has been conceptually simplified. The resultant equations can be solved separately, thus allowing the computations to be performed in parallel. The resultant formulas derived from the large change sensitivity approach have been presented. The equations, which allow this approach to be used to solve nonlinear circuits, have also been presented. A hierarchical approach for recombining the solutions was given. Finally this method was illustrated by a simple example.

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