EE 313: Basic EE
Chapter 4 Synopsis

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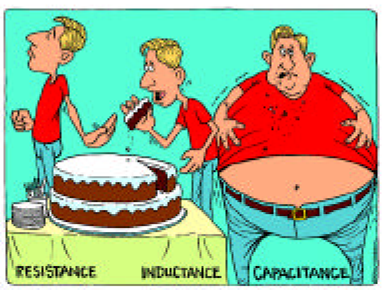
Chapter 4 Synopsis
Energy Storage Elements

4.1 The Capacitor
4.2 The Inductor
4.3 Continuity of Capacitor Voltages and Inductor Currents
4.4 Transformers and Coupled Coils
4.5 Circuit Differential Equations for Circuits Containing Energy Storage Elements
4.6 Response of Energy Storage Elements to DC Sources

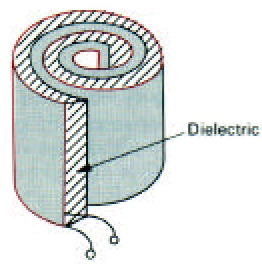
4.1 Energy Storage Elements: The Capacitor



R **L** **C**

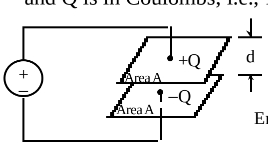
4.1 Energy Storage Elements: The Capacitor

FIGURE 4.2 Physical construction of a capacitor to occupy a small space.



4.1 Energy Storage Elements: The Capacitor

- Voltage V (in Volts) applied to the conceptual capacitor shown causes a charge Q in Coulombs to be deposited on the top plate
- The top plate's charge pushes an equal amount of (displacement) charge off of the bottom plate, leaving a negative (absence of) charge −Q on the bottom plate
- The relationship between the applied voltage and charge is Q = CV (C is in Farads when V is in Volts and Q is in Coulombs; i.e., 1 F = 1 Coulomb per Volt)



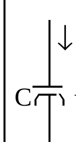
$$Q = CV$$

$$C = 8.84 \times 10^{-12}\ (A/d)$$

Energy is stored in the electric field existing between the plates

4.1 The Capacitor cont.

- Q(t) = Cv(t) , where C is constant
- Differentiate both sides with respect to time to obtain dQ(t) /dt = C × dv(t)/dt
- However, dQ(t)/dt = i(t) (current) ∴ i(t) = C × dv(t)/dt, where the passive sign convention is built into the formula (see below)



$$i(t) = C\frac{dv(t)}{dt} \quad \text{and}$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = \frac{1}{C}\int_{-\infty}^{t_0} i(\tau)d\tau + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau$$

$$= v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau$$

4.1 The Capacitor cont.

- Stored energy = $W(t) = CV^2/2$ (see below)

$$W(t) = \int_{-\infty}^{t} p(\tau)d\tau = \int_{-\infty}^{t} v(\tau)i(\tau)\,d\tau$$

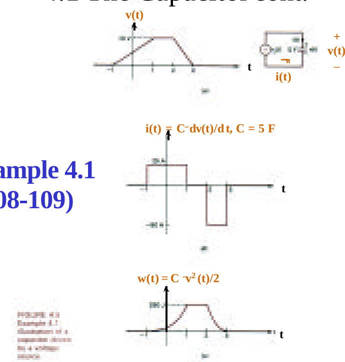
$$= \int_{-\infty}^{t} v(\tau)\,C\,\frac{dv(\tau)}{d\tau}\,d\tau$$

$$= C\int_{v(-\infty)}^{v(t)} v(\tau)dv(\tau) = \frac{1}{2}Cv(t)^2 - \frac{1}{2}Cv(-\infty)^2$$

$$\therefore\quad W(t) = \frac{1}{2}Cv(t)^2 \quad \text{(Assuming no initial stored energy)}$$

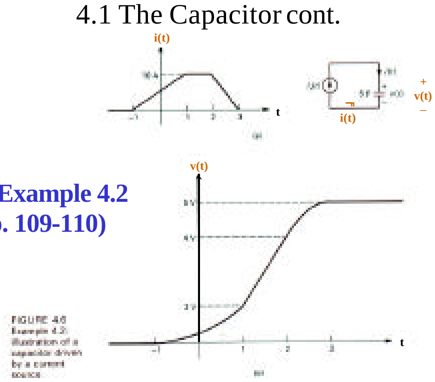
4.1 The Capacitor cont.

Text Example 4.1 (pp. 108-109)



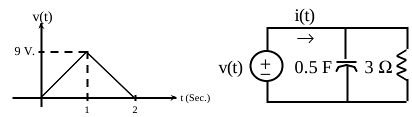
4.1 The Capacitor cont.

Text Example 4.2 (pp. 109-110)



4.1 The Capacitor cont.

Example 4-1

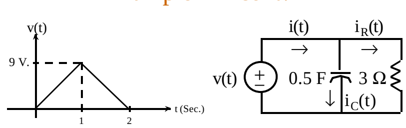


Sketch the source current i(t) and calculate the following:

A. The capacitor's maximum *stored* energy
B. The *total* energy consumed by the resistor
C. The *total* energy supplied by the source
D. The total charge that flows through the resistor
E. The resistor's average voltage

4.1 The Capacitor cont.

Example 4-1 cont.



$$v(t) = \begin{cases} 9t & \text{V., } 0 \le t \le 1 \text{ sec.} \\ -9t + 18 & \text{V., } 1 \le t \le 2 \text{ sec.} \\ 0 & \text{V., Elsewhere} \end{cases}$$

$i_R(t) = v(t)/3\Omega$ and $i_C(t) = 0.5F \times dv(t)/dt$

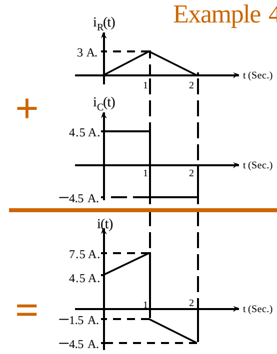
Therefore

$$i_R(t) = \begin{cases} 3t & \text{A., } 0 \le t \le 1 \text{ sec.} \\ -3t + 6 & \text{A., } 1 \le t \le 2 \text{ sec.} \\ 0 & \text{A., Elsewhere} \end{cases} \quad \text{and} \quad i_C(t) = \begin{cases} +4.5 & \text{A., } 0 < t < 1 \text{ sec.} \\ -4.5 & \text{A., } 1 < t < 2 \text{ sec.} \\ 0 & \text{A., Elsewhere} \end{cases}$$

and KCL yields: $i(t) = i_R(t) + i_C(t)$

4.1 The Capacitor cont.

Example 4-1 cont.



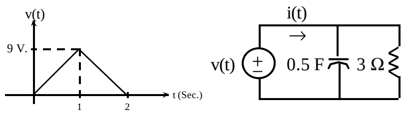
$$i_R(t) = \begin{cases} 3t & \text{A., } 0 \le t \le 1 \text{ sec.} \\ -3t + 6 & \text{A., } 1 \le t \le 2 \text{ sec.} \\ 0 & \text{A., Elsewhere} \end{cases}$$

$$i_C(t) = \begin{cases} +4.5 & \text{A., } 0 < t < 1 \text{ sec.} \\ -4.5 & \text{A., } 1 < t < 2 \text{ sec.} \\ 0 & \text{A., Elsewhere} \end{cases}$$

$$i(t) = \begin{cases} 3t + 4.5 & \text{A., } 0 < t < 1 \text{ sec.} \\ -3t + 1.5 & \text{A., } 1 < t < 2 \text{ sec.} \\ 0 & \text{A., Elsewhere} \end{cases}$$

4.1 The Capacitor cont.

Example 4-1 cont.

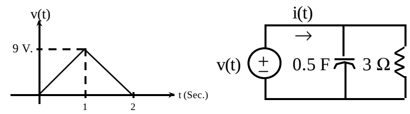


$$W_C = \frac{1}{2} C \times v^2$$

So the capacitor's maximum *stored* energy is $0.5 \times C \times (v_{max.})^2 = 0.5 \times 0.5\text{ F} \times (9\text{ V.})^2 = 20.25$ J.

4.1 The Capacitor cont.

Example 4-1 cont.

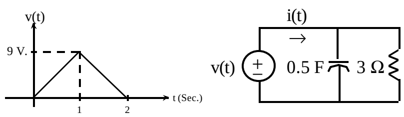


$$W_R = \int_{-\infty}^{\infty} p_R(t)dt = \int_0^2 v(t)^2/3\Omega\, dt$$

$$= \int_0^1 (9t\text{ V.})^2/3\Omega\, dt + \int_1^2 (-9t + 18\text{ V.})^2/3\Omega\, dt = 18\text{ J.}$$

4.1 The Capacitor cont.

Example 4-1 cont.



The capacitor can neither consume nor create energy— it can only *store* energy

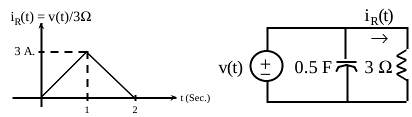
The voltage is nil for $0 \geq t \geq 2$ seconds so there is no net *stored* energy in the capacitor over the time interval $0 \leq t \leq 2$ seconds

Therefore it follows that the energy supplied (produced) by the source equals the energy consumed by the resistor (which is 18 J.)

What is the temporal behavior of the capacitor's stored energy?

4.1 The Capacitor cont.

Example 4-1 cont.



$$Q_R = \int_{-\infty}^{\infty} i_R(t)dt = \int_0^2 v(t)/3\Omega dt$$

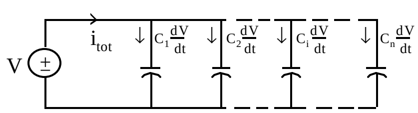
$$= \int_0^1 (3t\text{ A.})dt + \int_1^2 (-3t + 6\text{ A.})dt = 3\text{ C.}$$

and $V_{ave.} = W_R/Q_R = 18\text{ J.} / 3\text{ C.} = 6\text{ J./C.} = 6\text{ V.}$

4.1 The Capacitor cont.

- Capacitance in parallel adds:

  $C_{tot} = C_1 + C_2 + \ldots + C_n$

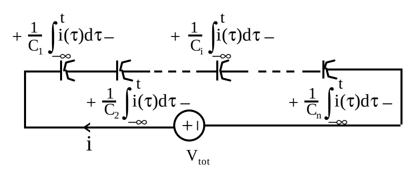


$$\text{KCL: } i_{tot} = \sum_{j=1}^{j=n} C_j \frac{dV}{dt} = \frac{dV}{dt} \sum_{j=1}^{j=n} C_j = C_{tot} \frac{dV}{dt}$$

4.1 The Capacitor cont.

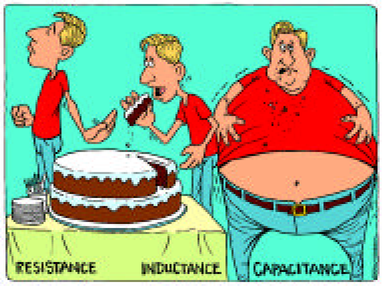
- Capacitance in series adds reciprocally:

  $C_{tot}^{-1} = C_1^{-1} + C_2^{-1} + \ldots + C_n^{-1}$



$$\text{KVL: } V_{tot} = \sum_{j=1}^{j=n} \frac{1}{C_j} \int_{-\infty}^{t} i(\tau)d\tau = \frac{1}{C_{tot}} \int_{-\infty}^{t} i(\tau)d\tau$$

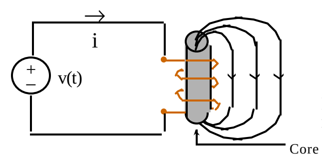
4.2 Energy Storage Elements: The Inductor



R **L** **C**

4.2 Energy Storage Elements: The Inductor

- Energy is stored in the magnetic field created by the current flowing in the coil (wrapped around a Ferromagnetic core)

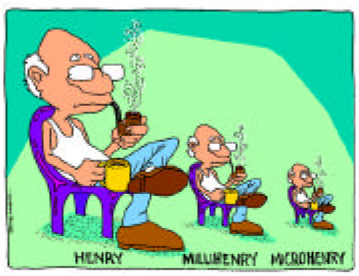


$$\Psi = Li$$

$$v(t) = \frac{d\Psi}{dt}$$

Energy is stored in the magnetic field near the core

4.2 Energy Storage Elements: The Inductor

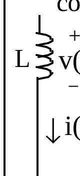


The Unit of Inductance (L) is the Henry

4.2 The Inductor cont.

$\psi = Li(t)$, where L is constant

- Differentiate both sides with respect to time to obtain $d\psi(t)/dt = L \times di(t)/dt$
- However, $d\psi(t)/dt = v(t)$ (voltage) ∴ $v(t) = L \times di(t)/dt$, where the passive sign convention is built into the formula (see below)



$$v(t) = L\frac{di(t)}{dt} \quad \text{and}$$

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau = \frac{1}{L}\int_{-\infty}^{t_0} v(\tau)d\tau + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau$$

$$= i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau$$

4.2 The Inductor cont.

- Stored energy = W(t) = $Li^2/2$ (see below)

$$W(t) = \int_{-\infty}^{t} p(\tau)d\tau = \int_{-\infty}^{t} v(\tau)i(\tau)\,d\tau$$

$$= \int_{-\infty}^{t} L\frac{di(\tau)}{d\tau}\,i(\tau)\,d\tau$$

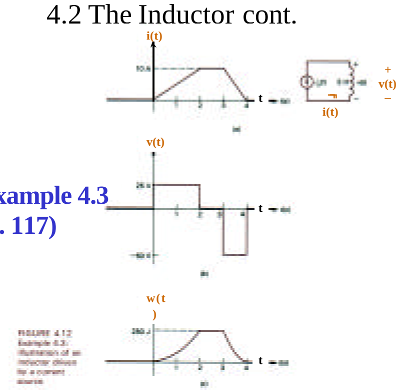
$$= L\int_{i(-\infty)}^{i(t)} i(\tau)\,di(\tau) = \frac{1}{2}Li(t)^2 - \frac{1}{2}Li(-\infty)^2$$

$$\therefore \quad W(t) = \frac{1}{2}Li(t)^2$$

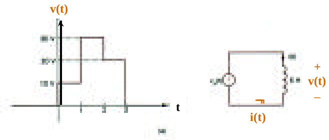
(Assuming no initial stored energy)

4.2 The Inductor cont.

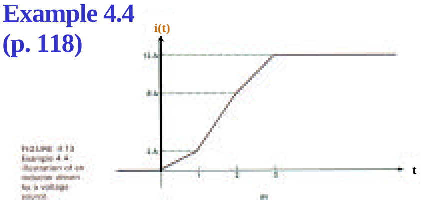
Text Example 4.3 (p. 117)



4.2 The Inductor cont.



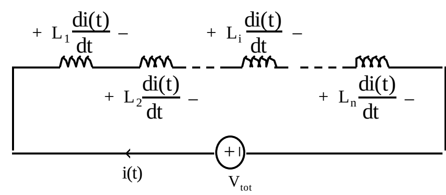
Text Example 4.4 (p. 118)



4.2 The Inductor cont.

- Inductance in series adds:

  $L_{tot} = L_1 + L_2 + \ldots + L_n$

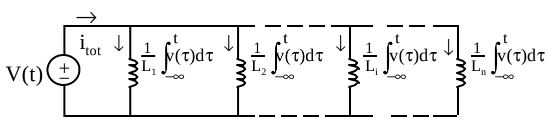


$$\text{KVL: } V_{tot} = \sum_{j=1}^{j=n} L_j \frac{di(t)}{dt} = L_{tot} \frac{di(t)}{dt}$$

4.2 The Inductor cont.

- Inductance in parallel adds reciprocally:

  $L_{tot}^{-1} = L_1^{-1} + L_2^{-1} + \ldots + L_n^{-1}$



$$\text{KCL: } i_{tot} = \sum_{j=1}^{j=n} \frac{1}{L_j} \int_{-\infty}^{t} v(\tau)d\tau = \frac{1}{L_{tot}} \int_{-\infty}^{t} v(\tau)d\tau$$

4.3 Continuity of Capacitor Voltage and Inductor Current

$$i(t) = C\frac{dv(t)}{dt}$$

C, + v(t) –

Finite i(t) requires finite $\frac{dv(t)}{dt}$

Finite $\frac{dv(t)}{dt}$ requires continuous v(t)

---

i(t)

$$v(t) = L\frac{di(t)}{dt}$$

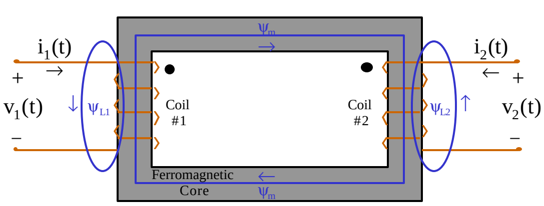
L, + v(t) –

Finite v(t) requires finite $\frac{di(t)}{dt}$

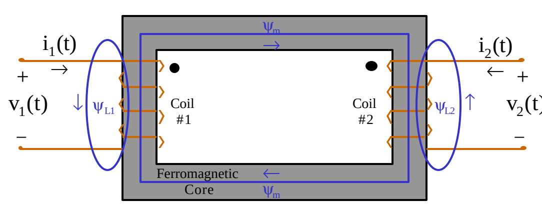
Finite $\frac{di(t)}{dt}$ requires continuous i(t)

4.4 Transformers and Coupled Coils

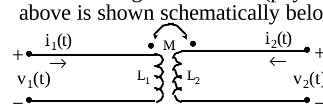
- Coil currents produce two types of magnetic flux (leakage & mutual)
- The (black) "dot" convention is used to indicate the configuration of how each coil is wound on the core



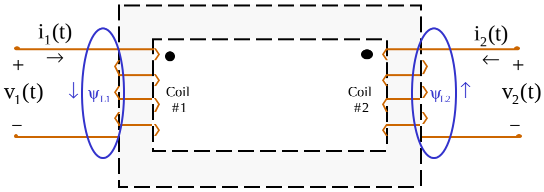
4.4 Transformers and Coupled Coils cont.



The coil-core arrangement shown (physically) above is shown schematically below



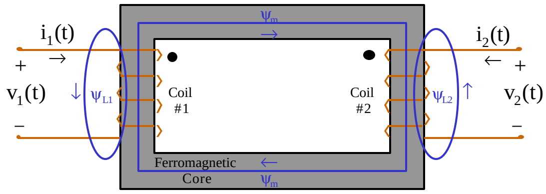
4.4 Transformers and Coupled Coils cont.



Without mutual induction, the coils are merely two independent inductors having the *decoupled* equations given below

$$v_1(t) = L_1 \frac{di_1(t)}{dt} \quad \text{and} \quad v_2(t) = L_2 \frac{di_2(t)}{dt}$$

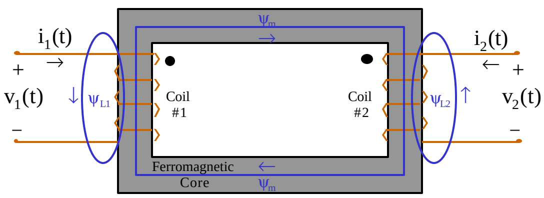
4.4 Transformers and Coupled Coils cont.



With mutual induction, the coils become dependent inductors having the *coupled* equations given below

$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt} \quad \text{and} \quad v_2(t) = L_2 \frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}$$

4.4 Transformers and Coupled Coils cont.

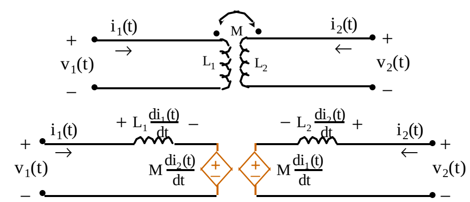


Each term in the KVLs below is a voltage

$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt} \quad \text{and} \quad v_2(t) = L_2 \frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}$$

4.4 Transformers and Coupled Coils cont.

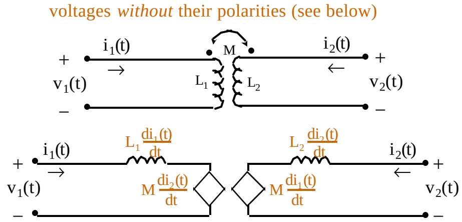
The voltage induced in each coil by the other coil's field can be represented using *dependent* voltage sources as shown below



KVL: $v_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}$ KVL: $v_2(t) = L_2\frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}$

4.4 Transformers and Coupled Coils cont.

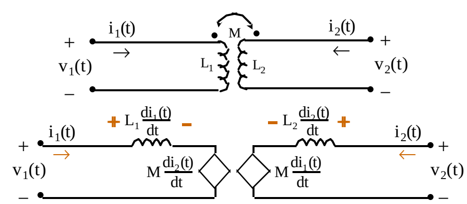
The lumped-circuit model can be developed in stages as follows:
Step 1: Draw the circuit showing self- and mutual-inductance voltages *without* their polarities (see below)



N.B.: The terminal voltages and currents *are* included in this first step's schematic but the polarities of the four voltages associated with the transformer *aren't*

4.4 Transformers and Coupled Coils cont.

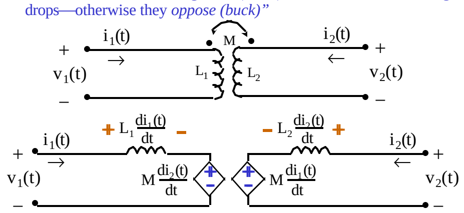
Step 2: Assign self-inductance voltage polarities using the passive sign convention in conjunction with the directions of $i_1(t)$ and $i_2(t)$



N.B.: $i_1(t)$ flows into the positive sign of the $L_1(di_1(t)/dt)$ voltage drop and $i_2(t)$ flows into the positive sign of the $L_2(di_2(t)/dt)$ voltage drop

4.4 Transformers and Coupled Coils cont.

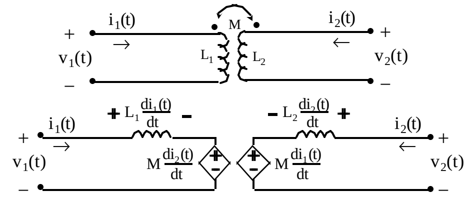
Step 3: Assign the (mutual inductance) dependent voltage source polarities using the rule: "If *both* currents flow *into* the dots *or* if *both* currents flow *out of* the dots, then the mutual voltages *aid (boost)* the self-inductance voltage drops—otherwise they *oppose (buck)*"



N.B.: Here, *aid (boost)* applies since *both* currents flow from the terminals (ports) *into* transformer dots

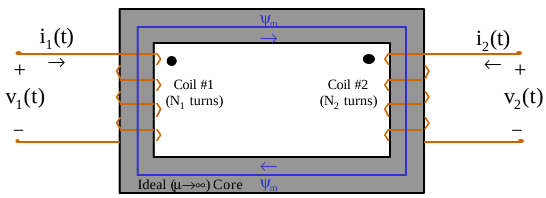
4.4 Transformers and Coupled Coils cont.

The transformer, its corresponding lumped-circuit (dependent-sources based) model and associated KVLs are shown below

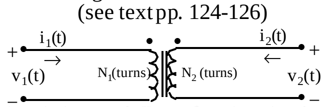


KVL: $v_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}$ KVL: $v_2(t) = L_2\frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}$

4.4 Ideal Transformer



If the core is a *perfect* conductor of (magnetic) flux, there is no leakage flux and the transformer is ideal (see text pp. 124-126)

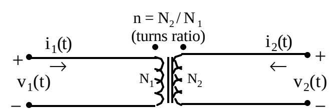


4.4 Ideal Transformer Properties

$v_2(t) = nv_1(t)$ and $i_1(t) = -ni_2(t)$
so that …

$p_1(t) = v_1(t)\, i_1(t) = -v_2(t) i_2(t) = -\, p_2(t)$

(i.e., $p_1(t) + p_2(t) = 0$ ∴ *lossless*)

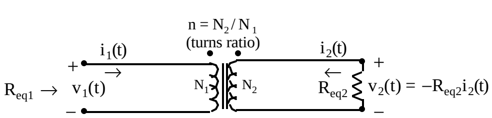


4.4 Ideal Transformer Properties cont.

$v_2(t) = nv_1(t)$ and $i_1(t) = -ni_2(t)$
so that …

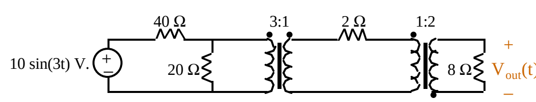
$R_{eq2} = -v_2(t)/i_2(t) = n^2(v_1(t)/i_1(t)) = n^2R_{eq1}$

(i.e., *reflected* impedance concept—see text Example 4.6, pp. 127-128)



4.4 Ideal Transformer Properties cont.

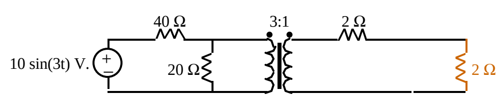
Example 4-2 (Text Problem 4.25)



Find $V_{out}(t)$

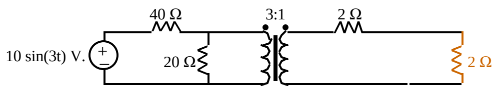
Solution

Begin by reflecting the 8 Ω resistor across the right transformer using $R_{eq} = (1/2)^2 \times 8\ \Omega = 2\ \Omega$ to obtain the equivalent circuit below

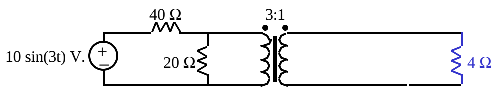


4.4 Ideal Transformer Properties cont.

Example 4-2 cont.

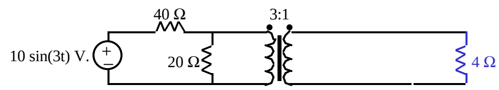


Next add the two series resistors (2 Ω + 2 Ω) = 4 Ω to obtain the circuit below

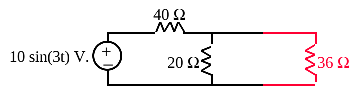


4.4 Ideal Transformer Properties cont.

Example 4-2 cont.

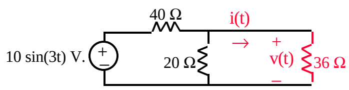


Reflect the 4 Ω resistor across the transformer using $R_{eq} = (3/1)^2 \times 4\ \Omega = 36\ \Omega$ to obtain the equivalent circuit below



4.4 Ideal Transformer Properties cont.

Example 4-2 cont.



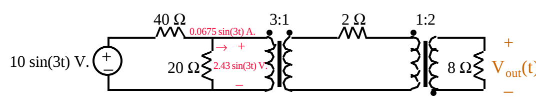
$20\ \Omega \parallel 36\ \Omega \approx 12.86\ \Omega$ along with voltage division yields:

$v(t) \approx [12.86\ \Omega / (40 + 12.86)\ \Omega] \times 10 \sin(3t)$ V. $\approx 2.43 \sin(3t)$ V.

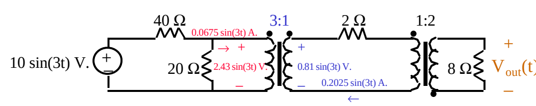
Then by Ohm's Law, $i(t) = v(t) / 36\ \Omega \approx 0.0675 \sin(3t)$ A.

4.4 Ideal Transformer Properties cont.

Example 4-2 cont.

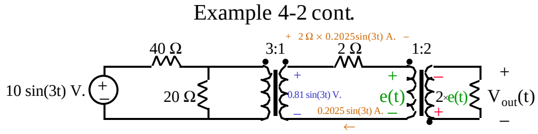


v(t) and i(t) as shown on the original schematic above, are next reflected to the right side of the left transformer as indicated below



4.4 Ideal Transformer Properties cont.

Example 4-2 cont.



<u>KVL</u>: $e(t) = 0.81 \sin(3t)$ V. $- 2\ \Omega \times 0.2025 \sin(3t)$ A. $\approx 0.405 \sin(3t)$ V.

As a result of the right transformer's turns ratio *and* the arrangement of its polarity dots (and KVL), the relationship between $V_{out}(t)$ and $e(t)$ is:

$V_{out}(t) = -2 \times e(t)$

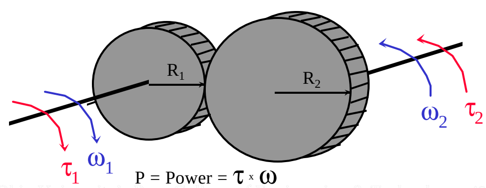
$\therefore\ V_{out}(t) \approx -0.81 \sin(3t)$ V.

4.4 Ideal Transformer Properties cont.

Gear analogy

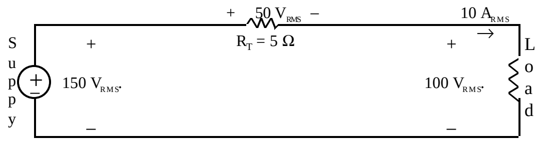
Transformer: $v_2(t) = nv_1(t)$ and $i_1(t) = -ni_2(t)$
where n = coil turns ratio (N_2 / N_1)

Gears: $\tau_2(t) = -n\tau_1(t)$ and $\omega_1(t) = -n\omega_2(t)$
where n = gear radii ratio (R_2 / R_1)



4.4 Power Transformer Application: Electric Power Transmission Application

Case 1: Load supplied directly from source via a transmission line



$P_{loss} = (10A_{RMS})^2 \times 5\Omega = 0.5$ kW (Half the load's power!)

$P_{load} = 100V_{RMS} \times 10A_{RMS} = 1$kW and $P_{Supply} = 150V_{RMS} \times 10A_{RMS} = 1.5$ kW

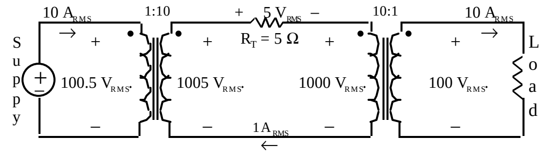
$\therefore \eta_{\%} = (1\text{kW} / 1.5\text{kW}) \times 100\% \approx 66.7\%$

The voltage drop between the supply and the load is $150\ V_{RMS} - 100\ V_{RMS} = 50\ V_{RMS}$

(A third of the supply's voltage!)

4.4 Power Transformer Application: Electric Power Transmission Application cont.

Case 2: Load supplied from source via a transmission line and transformers



$P_{loss} = (1A_{RMS})^2 \times 5\Omega = 5$ W. (Only 0.5% of the load's power!)

$P_{load} = 100V_{RMS} \times 10A_{RMS} = 1$kW and $P_{Supply} = 100.5V_{RMS} \times 10A_{RMS} = 1.005$ kW.

$\therefore \eta_{\%} = (1\text{kW} / 1.005\text{kW})\text{x}100\% \approx 99.5\%$

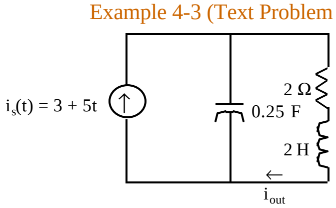
The *effective* voltage drop between the supply and the load is $100.5\ V_{RMS} - 100\ V_{RMS} = 0.5\ V_{RMS}$

(Only about 0.5% of the supply's voltage!)

4.5 Circuit Differential Equations

With the addition of energy storage elements, the formerly algebraic circuit equations become differential equations For example, consider text Problem 4.33 (p. 143) whose schematic is shown below—where the objective is to get the differential equation relating i_{out} to i_s

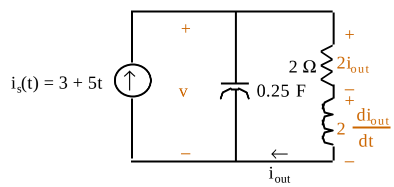
Example 4-3 (Text Problem 4.33)



4.5 Circuit Differential Equations cont.

After writing the voltage-current relationships on the schematic as shown below, KVL yields …

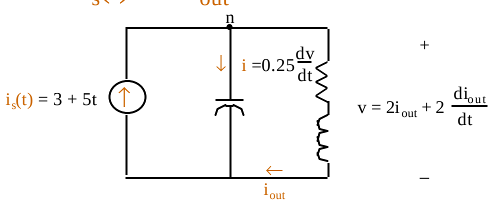
$$v = 2(di_{out}/dt) + 2i_{out}$$



4.5 Circuit Differential Equations cont.

After writing the voltage-current relationship for the capacitor on the schematic as shown below, KCL at node n yields …

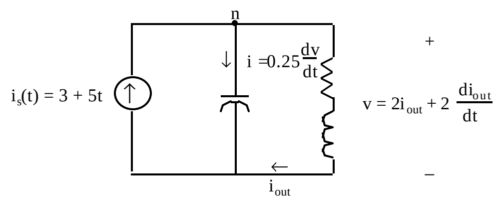
$$i_s(t) - i - i_{out} = 0\ \text{A. or ...}$$



4.5 Circuit Differential Equations cont.

$$i_s(t) - i - i_{out} = 0\ \text{A. or ...}$$

$$(3+5t) - 0.25\frac{dv}{dt} - i_{out} = 0\ \text{A.}$$

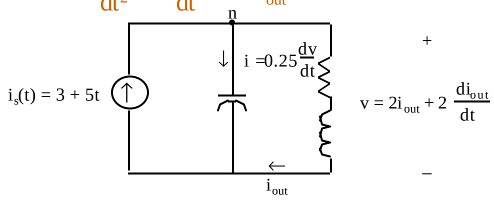


4.5 Circuit Differential Equations cont.

$(3+5t) - 0.25\frac{dv}{dt} - i_{out} = 0$ A.

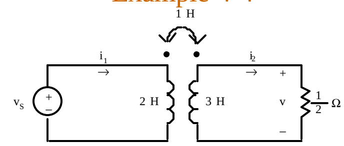
$(3+5t) - 0.25\frac{d}{dt}[2i_{out} + 2\frac{di_{out}}{dt}] - i_{out} = 0$ A.

$$\frac{d^2 i_{out}}{dt^2} + \frac{di_{out}}{dt} + 2i_{out} = 6 + 10t$$



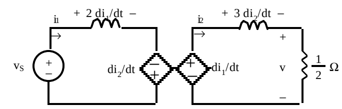
4.5 Circuit Differential Equations cont.

Example 4-4



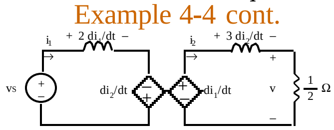
Determine the differential equation relating $v(t)$ to $v_S(t)$.

The equivalent circuit is



4.5 Circuit Differential Equations cont.

Example 4-4 cont.



KVL (left mesh):	$v_S - 2\,(di_1/dt) + di_2/dt = 0$	(A.)
KVL (right mesh):	$di_1/dt - 3\,di_2/dt - v = 0$	(B.)
Ohm's Law:	$v = 0.5\ \Omega \times i_2$	(C.)
Differentiate (C.) to obtain:	$di_2/dt = 2\,dv/dt$	(D.)
Substituting (D.) into (B.) yields:	$di_1/dt = 6\,dv/dt + v$	(E.)

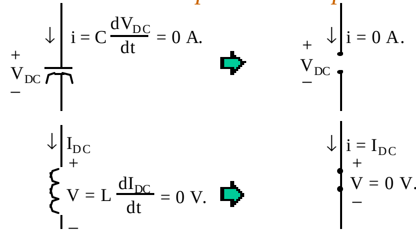
Substituting (D.) & (E.) into (A.) yields:

$$dv/dt + 0.2\,v = 0.1\,v_S$$

4.6 Energy Storage Elements Under DC Steady-State Conditions

Bottom Line

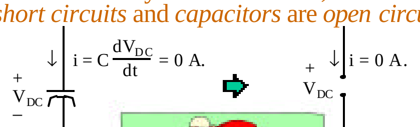
Under DC steady-state conditions, *inductors* are *short circuits* and *capacitors* are *open circuits*



4.6 Energy Storage Elements Under DC Steady-State Conditions

Bottom Line

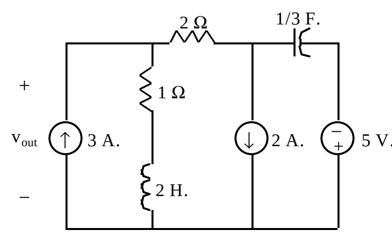
Under DC steady-state conditions, *inductors* are *short circuits* and *capacitors* are *open circuits*



4.6 Energy Storage Elements Under DC Steady-State Conditions cont.

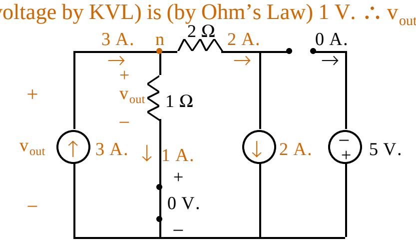
Example 4-5 (Text Problem 4.37, p. 144)

Find v_{out} under DC steady-state conditions



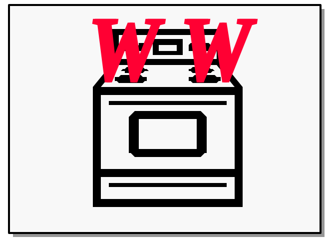
4.6 Energy Storage Elements Under DC Steady-State Conditions cont.

With the capacitor opened and the inductor shorted, the current flowing down through the 1 Ω resistor is 1 A. (by KCL at node n) so v_{out} (which is the resistor's voltage by KVL) is (by Ohm's Law) 1 V. ∴ v_{out} = 1 V.

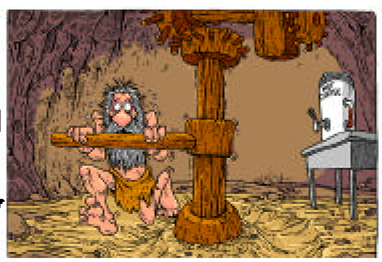


Examination

Identify the EE tune (*theme song*) schema below



Homework



1O5 Problems ∴ ≈ 11 per week

Teamwork!

Exam Payoff

Acknowledgement:

Cartoons courtesy of

We welcome your questions with Enthusiasm!!