Proceedings of the 2006 IEEE International Conference on Robotics and Automation
Orlando, Florida - May 2006

Disturbance Robustness Measures for Underconstrained Cable-Driven Robots

Paul Bosscher
Department of Mechanical Engineering
Ohio University
Athens, Ohio 45701
Email: bosscher@ohio.edu

Imme Ebert-Uphoff
School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, GA 30332
Email: ebert@me.gatech.edu

***Abstract*— This paper investigates the robustness of underconstrained cable-driven robots to external disturbance wrenches (force/moment combinations). Two cases are considered: impulsive disturbance wrenches and static (constant) disturbance wrenches. In the analysis of these cases a mapping to an intermediate space is introduced, which allows a restatement of Gauss' Principle of Least Constraint in a simpler form. Two measures are developed: the impulsive disturbance robustness measure and the static disturbance robustness measure. It is then shown that these two measures can be expressed as a single disturbance robustness measure, $\mathcal{R}$. This single measure enables underconstrained cable robots to be designed for maximum robustness to both impulsive and static disturbances.**

I. Introduction

Cable-driven robots, referred to as cable robots in this paper, are a type of robotic manipulator that has recently attracted interest for large workspace manipulation tasks. Cable robots are relatively simple in form, with multiple cables attached to a mobile platform or end-effector. The end-effector is manipulated by motors that can extend or retract the cables. These motors may be in fixed locations or mounted to mobile bases.

These robots are very versatile due to their stationary heavy components, few moving parts, large workspaces, transportability, ease of disassembly/reassembly and reconfiguration, and economical construction. Consequently, cable robots have been used for a variety of applications, including handling of heavy payloads [1], haptics [2], camera positioning [3] and high-speed manipulation [4]. Fig. 1 shows the RoboCrane [1], a six-cable manipulator for use in tasks such as material handling and manufacturing operations. Fig. 2 shows the SkyCam [3], which is used to position a video camera in stadiums and arenas.

A cable robot can be classified as either *underconstrained* or *fully-constrained* (a.k.a. incompletely restrained and completely constrained, respectively [5]). A cable robot is underconstrained if it relies on gravity to determine the pose (position and orientation) of the end-effector, while it is fully-constrained if the pose of the end-effector is completely determined by the lengths of the cables. The manipulators in Figs. 1 and 2 are both underconstrained. It has been shown in [6] that for an end-effector with n degrees of freedom $n+1$ cables are required to fully constrain the end-effector, where the wrenches (force/moment combinations) exerted by the cables must have "vector closure" [7] (i.e. positively span the n-dimensional wrench space [8]). This requirement for full constraint is often difficult to obtain for large workspace cable robots due to the likelihood of cables interfering with the end-effector, surroundings and each other. Thus underconstrained cable robots are often more practical for large workspace tasks.

Because underconstrained cable robots rely on gravity to determine the pose of the end-effector, it is possible for the pose of the end-effector to be changed by the presence of external disturbances. In fact it can be seen that some poses of a manipulator are more easily disturbed than other poses of the same manipulator. Because disturbance of the end-effector is undesirable, we wish to study and quantify the robustness of underconstrained cable robots to external disturbances so that cable robots can be designed with maximal disturbance robustness. In this paper two scenarios for external disturbances are considered: impulsive disturbance wrenches and static disturbance wrenches. The analysis utilizes Gauss' Principle of Least Constraint, which is restated in a simpler form using the intermediate space introduced in this paper. The study of disturbance robustness then leads to a disturbance robustness measure $\mathcal{R}$, a single measure that captures the robustness of underconstrained cable robots to both impulsive and static external disturbance wrenches.

II. Literature Review

The robustness analysis presented here builds upon the preliminary work by the author in [9] and [10]. Disturbance robustness of cable robots has also been referred to as stability. In [11] a condition for stability of a spatial 3-cable crane was developed based on the curvature of the path of the center of gravity. However this approach does not develop an adequate quantification of stability for general cable robots because it does not appropriately handle the mixed-dimensions of the task space.

Some researchers have also pointed out the similarity of cable robots to multi-fingered grasps [4], [12], [13]. The similarity arises from the uni-directional forces exerted by cables and fixture contacts. Grasp stability has been studied by a number of researchers (e.g. [14], [15]). However, these studies have often included the effects of friction, soft fingers

Fig. 1. A scale model of the RoboCrane [1].

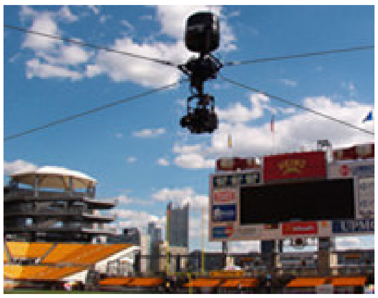


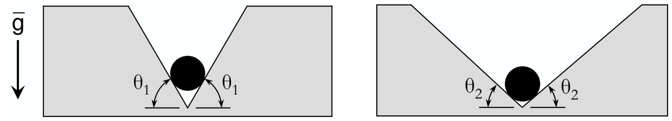
Fig. 2. The SkyCam (image from http://www.panasonic.com/business/provideo/news/news04_000.asp).

and curvature of the grasped object. Because a cable cannot exert forces perpendicular to the direction of the cable, there is no analogy to friction for cable robots[1]. Likewise, a cable cannot exert a moment about the axis along the cable, thus there is no analogy to soft finger contacts for cable robots. There is also no analogy in cable robots to the curvature of a grasped object [12]. The remaining studies of grasp stability typically focus on fixtures or grasps that fully constrain the object, which are analogous to fully constrained cable robots. Thus the majority of the research to date on fixture and grasp stability does not transfer easily to the disturbance robustness of underconstrained cable robots.

In the analysis of static disturbances this paper uses a construction called the Available Net Wrench Set, the set of all forces and moments that the manipulator can exert without violating cable tension limits. Similar concepts have been developed by other researchers, including the "capable force region" [16], the "set of manipulating forces" [17], the "closure domain" [18] and a "pseudo-pyramid" of wrenches [19]. In addition, the analysis of static disturbances presented here is similar to the analysis of disturbing and nondisturbing wrenches of a frictionless grasp in [20].

III. Disturbances

External disturbance wrenches[2] can take a variety of forms (e.g. impulsive, constant, cyclical, random) and can be applied



(a) High Slope Walls. (b) Low Slope Walls.

Fig. 3. Round Object Fixtured in a V-Block

to the manipulator at a variety of locations (e.g. at the end-effector, cables, motor mount locations). For the purpose of this analysis we will consider two of the more common disturbances: constant (static) external wrenches and impulsive external wrenches applied directly to the end-effector. Static disturbances are constant external wrenches applied to the end-effector by things such as a steady wind, a steady flow of water past an underwater end-effector, magnetic attraction, etc. Impulsive disturbance wrenches, in contrast, are brief impulses that impart a twist[3] (and corresponding kinetic energy) to the end-effector. Impulsive disturbance wrenches may be the result of a gust of wind or a collision between the end-effector and another object. These two types of disturbances are important to consider because many disturbances that a cable robot encounters can be modeled as one of these two types of disturbances.

Section IV examines the robustness of an underconstrained cable robot to impulsive disturbances. Following that, Section V examines the robustness of an underconstrained cable robot to static disturbances. For both analyses it is assumed that the cables have negligible mass and do not stretch or sag, the end-effector is a single rigid body with known cable attachment points on the end-effector relative to the center of gravity, and the locations of the attachments of the cables to the motors (or any pulleys the cables are routed through) are known. Cable lengths, the direction of gravity and the resulting pose of the end-effector are also assumed to be known. Additionally, the lengths of the cables are assumed to be held constant.

IV. Impulsive Disturbance Analysis

When an impulsive disturbance acts on the end-effector it imparts kinetic energy to the end-effector, causing it to swing away from its desired pose, typically causing cables to go slack. To reduce the effect of these disturbances we wish the end-effector to return to its original pose as quickly as possible, and thus we wish to maximize the restoring acceleration due to gravity. For the time being we will loosely define impulsive disturbance robustness as "the ability to quickly recover from an impulsive disturbance by returning to the original pose as quickly as possible."

As a simple illustration of this idea, consider Fig. 3. Here two identical round objects rest in two different V-blocks and each object is held in place by the frictionless contacts with the V-block walls and the influence of gravity. For the

[1] Multiple cables attached to a single point on the end-effector can approximate a frictional contact.

[2] A *wrench* is a force/moment combination. Using screw theory conventions a wrench is denoted $\$^w = \begin{pmatrix} \overline{F} \\ \overline{M} \end{pmatrix}$ where $\overline{F}$ and $\overline{M}$ are a force and moment acting at the center of gravity of the end-effector.

[3] A *twist* is a combination of linear and angular velocity. Using screw theory conventions a twist is denoted $\$^t = \begin{pmatrix} \overline{v} \\ \overline{\omega} \end{pmatrix}$ where $\overline{v}$ is the linear velocity of the center of gravity of the end-effector and $\overline{\omega}$ is the angular velocity of the end-effector.

sake of this example let us neglect the rotation of the objects and consider only translation. The fixtures do not provide complete constraint, thus the objects cannot resist any arbitrary disturbances. Consider the case where each of these objects are disturbed by an impulsive force such that they slide along the walls of the V-block, and each object is given the same initial velocity. The object in Fig. 3(a) will return to its original pose faster than the object in Fig. 3(b) because the V-block in Fig. 3(a) has steeper walls than the V-block in Fig. 3(b) (i.e. $\theta_1 > \theta_2$) causing a larger restoring acceleration of the object. Thus the fixture in Fig. 3(a) is more robust to this disturbance, as it ensures a faster return. As we will see, this concept can be extended to cable robots, as cables impose uni-directional constraints on the end-effector that locally are similar to V-block walls.

A. Initial Acceleration

The time it takes for the end-effector to return to its original pose is very difficult to compute because the nonlinear constraints imposed by the cables make the motion highly unpredictable and susceptible to erratic "bouncing." However, one of the key factors that determine the speed of return of the end-effector to its original pose is the *initial acceleration* of the end-effector back towards the original pose *immediately after the impulse has ended*. This acceleration can be determined analytically and is the dominant term for the response when the end-effector undergoes a small displacement as a result of the disturbance. Thus the initial acceleration of the end-effector after the impulse has ended will be used here to characterize the speed of disturbance recovery. In addition, the acceleration of the end-effector back to its original pose will depend on the direction in which the impulsive disturbance is applied. We will thus consider all possible initial accelerations of the end-effector and characterize the impulsive disturbance robustness of the pose by the worst case: *the smallest initial acceleration of the end-effector back to its original pose*.

After the disturbance has ended the only external wrench acting on the end-effector is the gravitational wrench, $\$^w_{grav} = \begin{pmatrix} m\overline{g} \\ \overline{0} \end{pmatrix}$, where m is the end-effector mass. The manipulator has ℓ translatinal degrees of freedom and k rotational degrees of freedom, thus $\overline{g}$ is the $\ell \times 1$ gravitational vector, directed downward and $\overline{0}$ is an $k \times 1$ zero vector. The motion of the end-effector is constrained by the cables that are taut. The acceleration of the end-effector at this point can be found using Gauss' Principle of Least Constraint [21], which states that for a constrained rigid body (the end-effector here), if an external wrench $\$^w_{ext}$ is applied to the body, the resulting acceleration $\$^a = \begin{pmatrix} \dot{\overline{v}} \\ \dot{\overline{\omega}} \end{pmatrix}$ of the body minimizes the following "acceleration energy":

$$\text{minimize } \frac{1}{2}(\$^w_{ext} - \boldsymbol{M}\$^a)^T \boldsymbol{M}^{-1}(\$^w_{ext} - \boldsymbol{M}\$^a) \qquad (1)$$
$$\text{subject to } \$^a \in \mathbb{A}$$

where $\boldsymbol{M}$ is the $n\times n$ generalized mass matrix of the body (n is the total number of degrees of freedom, where $n = k + \ell$):

$$\boldsymbol{M} = \begin{bmatrix} m\boldsymbol{I}^{\ell\times\ell} & \mathbf{0}^{\ell\times k} \\ \mathbf{0}^{\ell\times k} & \boldsymbol{H}^{k\times k} \end{bmatrix} \qquad (2)$$

where m is the end-effector mass, $\boldsymbol{I}^{\ell\times\ell}$ is the $\ell\times\ell$ identity matrix, $\mathbf{0}^{c\times d}$ is a $c\times d$ zero matrix and $\boldsymbol{H}$ is the angular inertia matrix.

$\mathbb{A}$ is the set of all *admissible accelerations*, the accelerations of the body (end-effector) that are consistent with the constraints imposed by the cables. Note that because we have assumed small displacements of the end-effector the velocities must be small as well, thus we assume $\overline{\omega}\times\boldsymbol{H}\overline{\omega} \ll \boldsymbol{H}\overline{\alpha}$. Thus the six-dimensional acceleration $\$^a$ becomes an *acceleration screw*, $\$^a = \begin{pmatrix} \overline{a} \\ \overline{\alpha} \end{pmatrix}$, where $\overline{a}$ and $\overline{\alpha}$ are the linear and angular acceleration of the end-effector, respectively.

B. Admissible Accelerations

We must now formulate $\mathbb{A}$, the set of admissible accelerations of the end-effector. Each cable i produces a constraint on the end-effector (that the distance from motor i to the point of attachment of cable i on the end-effector must be less than the length of cable i), which can be represented by a constraint surface in the n-dimensional configuration space of the end-effector. Utilizing our small displacement assumption, we will neglect the curvature of these constraint surfaces. As a result, the set of end-effector accelerations that are consistent with these constraints can be found in a straightforward way using the Jacobian relationship.

***Lemma* :**
Assuming small displacements and velocities, any admissible acceleration screw $\$^a$ must be a scalar multiple of an admissible twist $\$^t$.

***Proof* :**
Included in [10].

The set of *admissible twists*, $\mathbb{T}$, can be found using $\boldsymbol{J}^T$, the transpose of the Jacobian matrix.

$$\boldsymbol{J}^T = \left[\$^w_1 \;...\; \$^w_p\right] \qquad (3)$$

where $\$^w_i$ is the wrench from the i^{th} cable (in ray coordinates):

$$\$^w_i = \begin{pmatrix} \overline{u}_i \\ \overline{c}_i \times \overline{u}_i \end{pmatrix}. \qquad (4)$$

Here $\overline{u}_i$ is the unit vector running along cable i directed away from the end-effector, $\overline{c}_i$ is the vector from G, the center of gravity of the end-effector, to the point on the end-effector where cable i is connected, as illustrated in Fig. 4, and there are p cables attached to the end-effector. As shown in [9], [10] and [20], a twist is an admissible twist if and only if it has a non-negative dot product with all of the columns of $\boldsymbol{J}^T$:

$$\$^t \in \mathbb{T} \quad \Leftrightarrow \quad \$^t \cdot \$^w_i \geq 0, \quad i = 1, 2, ..., p. \qquad (5)$$

It can be shown that the set $\mathbb{T}$ is positively spanned by a set of *principal twists*, which is denoted $\mathbb{T}_P$. Note that $\mathbb{T}_P \subset \mathbb{T}$. In the case where $\boldsymbol{J}$ is full rank, each of the principal twists is

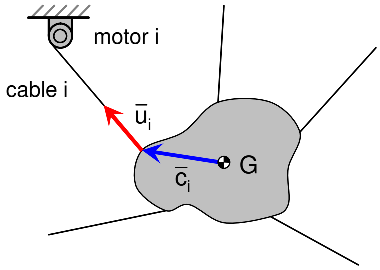


Fig. 4. Diagram of kinematic parameters.

reciprocal[4] to n-1 of the columns of $\boldsymbol{J}^T$. The principal twists, as we will see later, are key to formulating the disturbance robustness of a cable robot.

C. Intermediate Space

In order to solve for the initial acceleration of the end-effector after the impulse has ended, we will introduce a mapping that allows Gauss' Principle to be applied in a more straightforward manner. A linear mapping is presented here that maps twists, acceleration screws and wrenches to an *intermediate space* based on the inertial properties of the end-effector. This space has only linear units, allowing standard vector operations to be defined while at the same time producing results that retain physical significance due to the manner in which the mapping is defined. As a result, examining twists and wrenches in this space is useful for both the impulsive disturbance analysis and the static disturbance analysis in Section V.

Without loss of generality, the origin of the coordinate frame can be placed at the center of gravity, G, with the axes aligned with the principal axes of the end-effector[5]. A twist $\$^t$ can be mapped to a *generalized velocity*, $\hat{v}$, in the intermediate space by

$$\hat{v} = \boldsymbol{A}\$^t \tag{6}$$

where

$$\boldsymbol{A} = \left[\begin{array}{c|c} \boldsymbol{I}^{\ell\times\ell} & \boldsymbol{0}^{\ell\times k} \\ \hline \boldsymbol{0}^{k\times\ell} & \begin{matrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_k \end{matrix} \end{array}\right] \tag{7}$$

where ρ_i is the radius of gyration of the end-effector about axis i. Note that ($\hat{}$) is used to denote that the vector is defined in the intermediate space. Similarly, an acceleration screw $\$^a$ can be mapped to a *generalized acceleration*, $\hat{a}$, by

$$\hat{a} = \boldsymbol{A}\$^a. \tag{8}$$

A wrench $\$^w$ can be mapped to a *generalized force*, $\hat{f}$, by

$$\hat{f} = \boldsymbol{B}\$^w \tag{9}$$

where $\boldsymbol{B} = \boldsymbol{A}^{-1}$. Note that the mappings are chosen such that $m\boldsymbol{A}^2 = \boldsymbol{M}$ and $\frac{1}{m}\boldsymbol{B}^2 = \boldsymbol{M}^{-1}$.

[4]i.e. has a zero dot product.

[5]For the remainder of this paper all twists, wrenches and acceleration screws will be expressed in this coordinate frame.

These mappings produce n-dimensional vectors (usually $n = 3$ or $n = 6$), with consistent units of linear velocity, linear acceleration or force, respectively. This mapping uses the radii of gyration of the end-effector as characteristic lengths for the corresponding rotation elements of the twists, acceleration screws and wrenches. As a result, vector operations in the intermediate space have physical significance. First, the dot product between a generalized velocity and a generalized force is equal to the dot product between the associated twist and wrench:

$$\hat{v}\cdot\hat{f} \;=\; \hat{v}^T\hat{f} \;=\; {\$^t}^T\boldsymbol{A}^T\boldsymbol{B}\$^w \;=\; {\$^t}^T\$^w \;=\; \$^t\cdot\$^w. \tag{10}$$

As a result, perpendicularity between a generalized velocity and a generalized force in the intermediate space implies that the associated twist and wrench are reciprocal (they have a zero dot product).

Second, the square of the magnitude (Euclidean norm) of a generalized velocity $\hat{v}$ is proportional to the kinetic energy of the end-effector undergoing the associated twist $\$^t$. For compactness, denote the kinetic energy, KE, of a twist $\$^t$ (of the end-effector) as

$$KE\left(\$^t\right) = \frac{1}{2}{\$^t}^T\boldsymbol{M}\$^t. \tag{11}$$

Then

$$KE\left(\$^t\right) \;=\; \frac{m}{2}\,{\$^t}^T\boldsymbol{A}^2\$^t \;=\; \frac{m}{2}\parallel\hat{v}\parallel^2. \tag{12}$$

Third, the square of the magnitude of a generalized acceleration, $\hat{a}$ is proportional to the acceleration energy of the end-effector undergoing the associated acceleration $\$^a$. The acceleration energy, AE (also referred to as Appel's function, the Appelian or the Gibbs-Appel function [22]), of an acceleration $\$^a$ (of the end-effector) is

$$AE\left(\$^a\right) = \frac{1}{2}{\$^a}^T\boldsymbol{M}\$^a. \tag{13}$$

Then

$$AE\left(\$^a\right) \;=\; \frac{m}{2}\,{\$^a}^T\boldsymbol{A}^2\$^a \;=\; \frac{m}{2}\parallel\hat{a}\parallel^2. \tag{14}$$

Fourth, the square of the magnitude of a generalized force $\hat{f}$ is proportional to the acceleration energy of the resulting acceleration $\$^a$ the end-effector. This magnitude is defined here as the *acceleration energy norm* of a wrench. Using our small velocity assumption, the applied wrench is related to the resulting acceleration screw by:

$$\$^w = \boldsymbol{M}\$^a. \tag{15}$$

Noting that $\boldsymbol{M}$ is square, symmetric and invertible, we can see that

$$\$^a = \boldsymbol{M}^{-1}\$^w \tag{16}$$

thus

$$\begin{aligned} AE(\$^a) \;&=\; \frac{1}{2}{\$^w}^T\boldsymbol{M}^{-1}\boldsymbol{M}\boldsymbol{M}^{-1}\$^w \;=\; \frac{1}{2m}{\$^w}^T\boldsymbol{B}^2\$^w \\ &=\; \frac{1}{2m}{\$^w}^T\boldsymbol{B}^T\boldsymbol{B}\$^w \;=\; \frac{1}{2m}\parallel\hat{f}\parallel^2. \end{aligned} \tag{17}$$

We will refer to the acceleration energy of a wrench $\$^w$ with respect to the end-effector as

$$AE(\$^w) = \frac{1}{2}\$^{w^T}\boldsymbol{M}^{-1}\$^w. \tag{18}$$

The *acceleration energy norm* of a wrench, $\| \$^w \|_a$, is then defined as:

$$\begin{aligned} \| \$^w \|_a &= \sqrt{2m\, AE(\$^w)} && (19) \\ &= \sqrt{m\$^{w^T}\boldsymbol{M}^{-1}\$^w} && (20) \\ &= \| \hat{f} \| \,. && (21) \end{aligned}$$

While this may appear to be a new wrench norm, it is actually the basis of Gauss' Principle of Least Constraint (as we will see shortly) and as Subsection IV-D will discuss, is currently used (implicitly) in control of redundant manipulators, hybrid control schemes and manipulability measures. It is interesting to note that in the case where the end-effector is a point mass, the acceleration energy norm reduces to the Euclidean norm. Also, the acceleration energy norm of a wrench (with respect to a body) can also be directly related to the kinetic energy of the body after the wrench is applied for a small period of time [10].

In summary, twists, acceleration screws and wrenches can be mapped to an intermediate space, where all vectors have linear units and operations such as dot products and Euclidean norms have physical meaning. Twists and acceleration screws are mapped via the linear mapping $\boldsymbol{A}$ to generalized velocities and generalized accelerations, respectively. Wrenches are mapped via the linear mapping $\boldsymbol{B}$ to generalized forces.

D. Restatement of Gauss' Principle

Recall that Gauss' Principle of least constraint requires the minimization of what we called an "acceleration energy" term. Let the set of admissible accelerations, $\mathbb{A}$, be mapped via $\boldsymbol{A}$ to a set of *admissible generalized accelerations*, $\hat{\mathbb{A}}$:

$$\forall\ \hat{a},\ \text{ if }\ \hat{a} = \boldsymbol{A}\$^a\ \text{ then }\ \hat{a} \in \hat{\mathbb{A}}\ \Leftrightarrow\ \$^a \in \mathbb{A}. \tag{22}$$

Then mapping the elements within the "acceleration energy" term of Gauss' Principle to the intermediate space results in the following theorem:

***Theorem* :**
Gauss' Principle of Least Constraint can be restated as follows: for a constrained rigid body if an external wrench $\$^w_{ext}$ is applied to the body, then the resulting acceleration $\$^a = \begin{pmatrix} \dot{\overline{v}} \\ \dot{\omega} \end{pmatrix}$ of the body minimizes the acceleration energy norm of the reaction wrench (applied by the constraints):

$$\begin{aligned} &\text{minimize} \ \ \| \hat{f}_r \| && (23) \\ &\text{subject to } \hat{a} \in \hat{\mathbb{A}} \end{aligned}$$

where $\hat{a} = \boldsymbol{A}\$^a$ and $\hat{f}_r$ is the generalized reaction force:

$$\hat{f}_r = \hat{f}_{ext} - m\hat{a}. \tag{24}$$

where $\hat{f}_{ext} = \boldsymbol{B}\$^{ext}$.

***Proof* :**
Recall from (1) the original statement of Gauss' Principle of Least Constraint. Then

$$\begin{aligned} \frac{1}{2}(\$^w_{ext} - \boldsymbol{M}\$^a)^T\boldsymbol{M}^{-1}(\$^w_{ext} - \boldsymbol{M}\$^a) &= \\ \frac{1}{2m}(\$^w_{ext} - \boldsymbol{M}\$^a)^T\boldsymbol{B}^T\boldsymbol{B}(\$^w_{ext} - \boldsymbol{M}\$^a) &= \\ \frac{1}{2m}(\boldsymbol{B}\$^w_{ext} - m\boldsymbol{A}\$^a)^T(\boldsymbol{B}\$^w_{ext} - m\boldsymbol{A}\$^a) &= \\ \frac{1}{2m}(\hat{f}_{ext} - m\hat{a})^T(\hat{f}_{ext} - m\hat{a}) &= \\ \frac{1}{2m} \parallel \hat{f}_r \parallel^2 & \end{aligned} \tag{25}$$

where $\hat{f}_r$ is the generalized reaction force ($\hat{f}_r = \hat{f}_{ext} - m\hat{a}$) corresponding to the reaction wrench $\$^w_r$ applied to the end-effector by the constraints (i.e. the cables). Here $\hat{f}_r = \boldsymbol{B}\$^w_r$ and $\hat{f}_{ext} = \boldsymbol{B}\$^w_{ext}$. Then the minimization in (1) corresponds to a minimization of $\| \hat{f}_r \|$. □

Thus if an external wrench is applied to a constrained body, Gauss' Principle implies that the system responds by minimizing the *acceleration energy norm of the reaction wrenches* applied to the body by the constraints (cables here). Note that this theorem does *not* rely on the assumption of small displacements/velocities of the body and thus applies whenever the original form of Gauss' Principle applies.

Note that while acceleration energy is not a frequently used concept, the acceleration energy norm of wrenches has actually been used previously. In [23] it is shown that minimizing the acceleration energy norm of the joint torques in a redundant serial robot results in the well known choice of minimizing the instantaneous kinetic energy of the manipulator. Similarly, in [24] the inertia matrix $\boldsymbol{M}$ is used as a weighting matrix for decomposing twists and wrenches in a hybrid control scheme, which implicitly uses the acceleration energy norm to decompose wrenches. In [25] the inertia matrix is used in several redundancy-resolution schemes for serial robots, including the "Inertia-Inverse Weighted Torque Scheme," which minimizes the acceleration energy norm of the joint torques. In [26] the inertia matrix $\boldsymbol{M}$ is used as a weighting matrix for a manipulability measure, which also implicitly uses the acceleration energy norm for wrenches.

E. Impulsive Disturbance Robustness Measure

By considering Gauss' Principle in the intermediate space, it is very straightforward to find the smallest initial acceleration of the end-effector back towards its original pose. Here we will be defining the magnitude of an acceleration screw by the magnitude of the corresponding generalized acceleration:

$$\| \$^a \| \triangleq \| \hat{a} \| \qquad \text{where } \ \hat{a} = \boldsymbol{A}\$^a. \tag{26}$$

Next it is necessary to consider all possible scenarios for the impulsive disturbance (i.e. it can cause 1 cable to go slack, it can cause 2 cables to go slack, etc.) and apply the restatement of Gauss' Principle in each case. The examination of each scenario is not included here due to space limitations, but can be found in [10]. The result of this analysis is that the smallest acceleration of the end-effector back towards its original pose occurs when the disturbance moves the end-effector along a principal twist. Moreover, if the end-effector is disturbed along a principal twist $\$^t_P$, the acceleration of the end-effector back towards the original pose (after the impulse has ended) has magnitude $mg\cos\theta(\$^t_P)$, where $\theta(\$^t_P)$ is the angle between the generalized principal velocity $\hat{v}_P$ ($\hat{v}_P = \boldsymbol{A}\$^t_P$) and $-\hat{f}_{grav}$ (i.e. the vertical direction, $\hat{f}_{ext} = \hat{f}_{grav}$):

$$\theta(\$^t_P) = \cos^{-1}\left(\frac{-\hat{f}_{grav}\cdot\hat{v}_P}{\|\hat{f}_{grav}\cdot\hat{v}_P\|}\right) \tag{27}$$

where $\cos^{-1}$ is assumed to yield a result between 0 and π.

Definition: The *impulsive disturbance robustness*, $\mathcal{R}_i$ is defined as

$$\boxed{\mathcal{R}_i = \min_{\$^t\in\mathbb{T}_P} \cos\left(\theta(\$^t)\right)}. \tag{28}$$

Then if an impulsive disturbance is applied to the manipulator the smallest possible (i.e. worst case) initial acceleration of the end-effector is $\|\$^a\| = g\mathcal{R}_i$. It can also be shown that the corresponding vertical acceleration of the center of gravity of the end-effector is $a_{vert,min} = -g\mathcal{R}_i^2$.

This measure can take on values between 0 and 1, with 1 corresponding to a manipulator with the best/highest impulsive disturbance robustness (where the initial acceleration towards the original pose is $\overline{g}$) and 0 corresponding to the worst/lowest impulsive disturbance robustness (where the initial acceleration towards the original pose is 0). Note also that $\mathcal{R}_i$ is pose-dependent.

V. Static Disturbance Analysis

This section investigates the effects of static disturbance wrenches on the pose of an underconstrained cable robot. Because underconstrained cable robots cannot resist any arbitrary external wrench, there is a set of external wrenches that will disturb the end-effector. Here we will characterize the *static disturbance robustness* of the manipulator pose by the "smallest" static wrench that will disturb the end-effector. This smallest static disturbance wrench represents a sort of "worst-case scenario" for the manipulator, and can be thought of as being applied in the direction of least constraint for the end-effector.

A. Disturbance Wrenches

In order to perform this analysis, it is first necessary to determine the set of external wrenches that disturb the end-effector. In [27] it was shown how to find the set of wrenches that can be exerted by the manipulator on its surroundings. This set, termed the *Available Net Wrench Set* (abbreviated NW_{avail}), is defined as

$$NW_{avail} = \left\{\$^w : \$^w = \boldsymbol{J}^T\overline{t} + \$^w_{ext};\ t_{i,max} \geq t_i \geq 0\right\}, \tag{29}$$

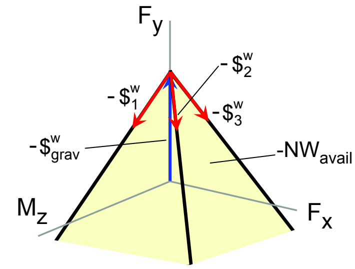


Fig. 5. Example $-NW_{avail}$ (where each $t_{i,max}$ is very large).

where $\overline{t}$ is the vector of cable tensions: $\overline{t} = (t_1\ t_2 \ldots t_p)$. In other words, NW_{avail} is the set of all wrenches that the end-effector can apply to its surroundings, taking into account the tension limits in the cables and the effect of constant external wrenches such as gravity. The restriction that $t_{i,max} \geq t_i \geq 0$ stems from the fact that each cable can pull but not push (i.e. a cable cannot have negative tension) and is restricted to be less than or equal to a maximum tension $t_{i,max}$. Because the manipulator can exert any wrench in NW_{avail}, it can counteract any externally applied wrench in $-NW_{avail}$, where

$$\$^w \in NW_{avail} \ \Leftrightarrow\ -\$^w \in -NW_{avail}. \tag{30}$$

Thus $-NW_{avail}$ is the set of external wrenches that can be resisted by the manipulator without violating the tension limits of the cables[6]. The boundaries of $-NW_{avail}$ represent the limits of the wrenches that can be resisted by the manipulator at the current pose. As an example, Fig. 5 shows an example $-NW_{avail}$ for a planar cable robot (which exerts wrenches of $\$^w = (F_x\ F_y\ M_z)^T$). Thus any external wrench applied to the manipulator that is not in the $-NW_{avail}$ shown in Fig. 5 cannot be resisted by the manipulator at its current pose.

Because we are interested in finding the smallest static disturbance wrench we only need to consider the set of wrenches that *just begin* to disturb the end-effector (i.e. the set of wrenches on the *boundaries* of $-NW_{avail}$). Let the set of all wrenches on the boundaries of $-NW_{avail}$ constitute the set of all *boundary wrenches*, denoted $\mathbb{B}$. That is, a wrench $\$^w_{bound}$ is defined as a boundary wrench (i.e. $\$^w_{bound} \in \mathbb{B}$) if and only if for $\alpha \in I\!R$,

$$\begin{aligned} & 1 \geq \alpha \geq 0 \ \Rightarrow\ \alpha\$^w_{bound} \in -NW_{avail} \\ \text{and} \quad & \alpha > 1 \ \Rightarrow\ \alpha\$^w_{bound} \notin -NW_{avail}. \end{aligned} \tag{31}$$

These wrenches are the boundary between the set of wrenches that can be resisted by the manipulator and the set of wrenches that can not. Thus the smallest static disturbance wrench must be in $\mathbb{B}$.

B. Wrench Norm

In order to find the smallest static disturbance wrench, it is necessary to define a wrench norm. Because wrenches include

[6]While it is important not to exceed upper tension limits in the cables, the upper tension limits are often very large and can typically be made irrelevant by appropriate sizing of the cables and motors and constraining the end-effector to operate in an appropriate workspace. Thus for the purpose of this analysis the upper tension limits will be ignored.

both forces and moments, the standard Euclidean norm is not defined. However, in Subsection IV-C we showed that the *acceleration energy norm* of a wrench is a physically meaningful norm for wrenches. While it may seem unusual to use dynamic properties of the end-effector to define a norm on static wrenches, this has actually been done previously in [24], where the inertia matrix of the end-effector was used as a weighting matrix for a hybrid control scheme to determine the (pseudo) static wrenches exerted by a robotic manipulator on its environment. Thus we will use the acceleration energy norm to define the magnitude of static disturbance wrenches.

C. Static Disturbance Robustness Measure

The proposed static disturbance robustness measure is based on the magnitude of the smallest static disturbance wrench, using the acceleration energy wrench norm.

Definition: The *static disturbance robustness* measure, $\mathcal{R}_s$, for a pose of a cable robot is:

$$\boxed{\mathcal{R}_s = \frac{1}{mg} \min_{\$^w \in \mathbb{B}} \sqrt{\$^{wT} \boldsymbol{B}^2 \$^w}} \quad (32)$$

$$= \frac{1}{mg} \min_{\$^w \in \mathbb{B}} \| \$^w \|_a \quad (33)$$

$$= \frac{1}{mg} \min_{\hat{f} \in \hat{\mathbb{B}}} \| \hat{f} \| . \quad (34)$$

where $\hat{\mathbb{B}}$ is the set of wrenches in $\mathbb{B}$ mapped to the intermediate space via $\boldsymbol{B}$.

This measure uses the acceleration energy norm to define the magnitude of a disturbance wrench, which is equivalent to mapping all wrenches to the intermediate space and considering the magnitude of the corresponding generalized forces. The measure then finds the smallest wrench in $\mathbb{B}$, which is the set of all boundary wrenches (the static wrenches that just begin to disturb the manipulator). The resulting magnitude is then normalized by the factor $\frac{1}{mg}$. This normalization produces a result where $\mathcal{R}_s$ can take on values between 0 and 1, with 1 corresponding to a manipulator pose with the highest possible static disturbance robustness and 0 corresponding to a manipulator pose with the lowest possible static disturbance robustness. The magnitude of this smallest static disturbance wrench is $mg\mathcal{R}_s$. Since $\mathcal{R}_s$ is pose-dependent, $\mathcal{R}_s$ will have different values throughout the workspace of a manipulator.

By providing a lower bound on the wrenches that disturb the end-effector, this measure also allows for other types of disturbances to be analyzed. For example, an oscillating disturbance wrench or random wrenches (like white noise) could be applied to the end-effector. If the acceleration energy norm of such a disturbance has a known upper bound that is less than or equal to $mg\mathcal{R}_s$ then the end-effector can resist that disturbance.

D. Calculating $\mathcal{R}_s$

Finding the smallest generalized force in $\hat{\mathbb{B}}$ in the intermediate space is fairly simple. Because $\mathbb{B}$ is made up of planar segments, $\hat{\mathbb{B}}$ is made up of planar segments (because the mapping to the intermediate space is linear). From geometry we know that the shortest distance between a point (the origin) and a plane (a segment of $\hat{\mathbb{B}}$) is along a line perpendicular to the plane. Thus $\hat{f}_{min}$, the smallest generalized force in $\hat{\mathbb{B}}$, will be perpendicular to one of these segments as shown in Fig. 6. In order to find the smallest generalized force only the forces perpendicular to the planar segments of $\hat{\mathbb{B}}$ need to be considered. Once $\hat{f}_{min}$ is found, it can be substituted into (34) to find the static disturbance robustness of the pose. A detailed numerical example of calculating $\mathcal{R}_i$ and $\mathcal{R}_s$ is included in [10], but is not included here due to space limitations.

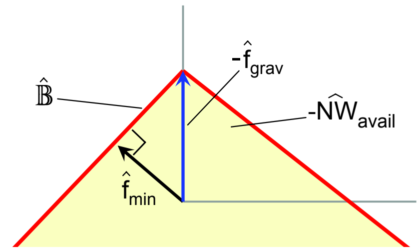


Fig. 6. Calculation of the smallest wrench in $\hat{\mathbb{B}}$.

VI. Disturbance Robustness Measure

Given these two robustness measures, $\mathcal{R}_i$ and $\mathcal{R}_s$, it is of interest to see how they relate to each other. The relationship between these measures can be seen through geometric analysis of NW_{avail} in the intermediate space. Although space limitations prevent the full description of this analysis (which can be found in [10]), it will be summarized briefly here.

A. Relating $\mathcal{R}_i$ and $\mathcal{R}_s$

Because each principal twist $\$^t_P$ is reciprocal to n-1 columns of $\boldsymbol{J}^T$, each corresponding generalized principal velocity $\hat{v}_P$ is perpendicular to one of the planar sides of $-\hat{NW}_{avail}$ and is thus perpendicular to one of the planar sides of $\hat{\mathbb{B}}$. Because $\hat{f}_{min}$ is also perpendicular to one of the planar sides of $\hat{\mathbb{B}}$, $\hat{f}_{min}$ must be parallel to a $\hat{v}_P$. In fact, it can be shown that $\hat{f}_{min}$ is parallel to the $\hat{v}_P$ that satisfies (28), corresponding to the smallest initial acceleration of the end-effector back towards its original pose. Through geometry it can then be shown that $\mathcal{R}_s = \cos(\theta(\$^t))$, where $\theta(\$^t)$ satisfies (28). Thus we see that $\mathcal{R}_i = \mathcal{R}_s$. Let us therefore combine these two measures and define the Disturbance Robustness Measure:

Definition: The *Disturbance Robustness Measure*, $\mathcal{R}$ is defined as:

$$\boxed{\mathcal{R} = \mathcal{R}_s = \mathcal{R}_i}. \quad (35)$$

Thus this single measure describes *both* the static and impulsive disturbance robustness of the manipulator.

B. Discussion

The fact that this measure captures both the static and impulsive disturbance robustness is very significant. In both cases the measure describes the worst-case scenario for the manipulator: the smallest static disturbance wrench ($\| \$^w_{min} \| = mg\mathcal{R}$) and the lowest acceleration of an impulsively disturbed end-effector back to its original pose ($\| \$^a_{min} \| = g\mathcal{R}$). Because this is an energy-based measure, it is both scale- and frame-invariant. The measure also handles redundant manipulators just as easily as it does non-redundant manipulators. In addition, it is not difficult to adapt this measure to allow robustness to be computed in the presence of any constant external wrench, not just gravity. This can be accomplished by simply replacing $\$^{w}_{grav}$ in all calculations with the net external wrench due to gravity and additional constant external wrenches.

It is important to understand that this measure only describes the robustness of a single pose of a manipulator. However, measures can be constructed using $\mathcal{R}$ that describe the overall measure of robustness for a manipulator as is done by the author in [10]. In addition, the robustness measure was extended to apply to cable robots with multi-body end-effectors [10].

A remaining challenge is that it is not obvious what an acceptable minimum value is for $\mathcal{R}$. This will typically be application-dependent and experimentation and practical considerations may need to be taken into account to determine what constitutes an appropriate minimum necessary value of $\mathcal{R}$ for a given manipulator.

VII. CONCLUSIONS

This paper examined the robustness of underconstrained cable robots to external disturbances. Two measures, the impulsive disturbance robustness measure, $\mathcal{R}_i$, and the static disturbance robustness measure, $\mathcal{R}_s$, were developed. These measures rely on a mapping to an intermediate space and characterizing the magnitude of an acceleration by the resulting acceleration energy of the body undergoing that acceleration. Using this intermediate space a simplified restatement of Gauss' Principle of Least Constraint was developed and the connection of this principle with the acceleration energy wrench norm was demonstrated. Lastly, it was shown that these two measures are the same, resulting in the definition of the disturbance robustness measure, $\mathcal{R}$. Thus if the end-effector is disturbed by an impulse the smallest initial acceleration of the end-effector back towards the original pose has magnitude $g\mathcal{R}$ and the smallest static wrench that begins to disturb the end-effector has magnitude $mg\mathcal{R}$. This measure now enables underconstrained cable robots to be designed for maximum robustness to impulsive and static disturbances.

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