ChE 400: Differential Equations: Analytical Solutions using Characteristic Polynomial Method (H-7a)

Gerardine G. Botte

Objectives

- General:
  - Classify Differential Equations
  - Apply the concepts of this chapter for the solution of chemical engineering problems that require the calculation of homogeneous-linear-ordinary differential equations
- Specific objectives:
  - Recognize linear from nonlinear differential equations
  - Recognize homogenous from non-homogeneous differential equations
  - Distinguish the order of a differential equations
  - Use the characteristic polynomial method to solve ODEs

Outline

- Differential Equations
  - Classification
  - Examples
- Ordinary Differential Equations (ODE)
  - Analytical Solution Methods
    - Characteristic Polynomial
    - Laplace Transform
- Lab Practice
  - Use of Matlab to calculate Laplace transform

Differential Equations
- Classification
- Exercises

ODE
- Analytical Solutions
  - Characteristic Polynomial

Differential Equations

- A differential equation is an equation involving an unknown function and its derivatives, example:

$$\frac{dy}{dx} = 5x + 3$$

Differential Equations
- Classification
- Exercises

ODE
- Analytical Solutions
  - Characteristic Polynomial

Methods of Solution

- Depend on the type of differential equation
- Analytical: only for linear differential equations
- Numerical: for linear and nonlinear differential equations

Differential Equations
- Classification
- Exercises

ODE
- Analytical Solutions
  - Characteristic Polynomial

Nomenclature

- y’, y”, y”’, $y^{(4)}$, …, $y^{(n)}$ are often used to represent, respectively, the first, second, third, fourth, …, nth derivatives of y with respect to the independent variable under consideration
- Thus:
  - y” represents d^2y/dx^2 if the independent variable is “x” but represents d^2y/dp^2 if the independent variable is “p”

Differential Equations

Differential Equations
– Classification
– Exercises
ODE
– Analytical Solutions
• Characteristic Polynomial

- Differential Equations can be classified according to:
  - Linearity
  - Order
  - Boundary Conditions (BC's)
  - Type (Variables involved)

According to Type (Variables Involved):

Differential Equations
– Classification
– Exercises
ODE
– Analytical Solutions
• Characteristic Polynomial

- Ordinary Differential Equations (ODE)
- Partial Differential Equations (PDE)

ODE

Differential Equations
– Classification
– Exercises
ODE
– Analytical Solutions
• Characteristic Polynomial

- The function involves one independent variable, example:

$$\frac{dy}{dx} = x^2$$

y: dependent variable

x: independent variable

Differential Equations
– **Classification**
– Exercises
ODE
– Analytical Solutions
 • Characteristic Polynomial

PDE

- The function involves two or more independent variables, example:

$$\frac{\partial c}{\partial t} = -D\frac{\partial^2 c}{\partial x^2}$$

c: dependent variable

t: independent variable

x: independent variable

Notice that the symbol for partial differentiation is used

Differential Equations
– **Classification**
– Exercises
ODE
– Analytical Solutions
 • Characteristic Polynomial

According to Order

- Order: is defined as the highest order of the derivatives
- Depending on the order the eqs. can be classified in: first order, second order, third order, etc, e.g.,:

$$\frac{dy}{dx} = x^2 \quad \text{First order}$$

Differential Equations
– **Classification**
– Exercises
ODE
– Analytical Solutions
 • Characteristic Polynomial

According to Order <u>Continued</u>

- More examples:

$$e^y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 1 \quad \text{Second order}$$

- The number of boundary conditions needed is equal to the order of the differential equation, e.g.,
  – First order differential equation needs one boundary condition

Differential Equations
– **Classification**
– Exercises
ODE
– Analytical Solutions
• Characteristic Polynomial

According to Linearity

- They can be classified into
  - Linear
  - Nonlinear

Differential Equations
– **Classification**
– Exercises
ODE
– Analytical Solutions
• Characteristic Polynomial

Linearity

- An nth-order ODE in the unknown function "y" and the independent variable "x" is linear if it has the form:

$$b_n(x)\frac{d^n y}{dx^n} + b_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + b_1(x)\frac{dy}{dx} + b_0(x)y = g(x) \qquad (1)$$

- The functions bj(x) (j=0, 1, 2..,n) and g(x) are known and depend only on the variable "x" (independent variable)
- If the differential equation can't be expressed in the form of Eq. 1, the equation is nonlinear

Differential Equations
– **Classification**
– Exercises
ODE
– Analytical Solutions
• Characteristic Polynomial

Linearity continued

- Notice that according to Eq.(1) the coefficients and the function g(x) do not depend on the dependent variable (linear ODE)
- In nonlinear differential equations the coefficients (b), depend on the dependent variable
- The first step to analyze linearity is to identify the dependent and independent variables

Linearity
continued

- Examples: Dependent variable

$$5x^2 \frac{dy}{dx} = 5x + 3 \quad \text{Linear}$$

$$e^y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 1 \quad \text{nonlinear}$$

Dependent variable

Linearity
Continued

- Linear differential equations can be classified into
  - Homogenous
    - G(x) in Eq. (1) is equal to zero, e.g.:

$$5x^2 \frac{dy}{dx} = 0$$

  - Non-homogenous
    - G(x) in Eq. (1) is not zero, e.g.:

$$5x^2 \frac{dy}{dx} = 5x + 3$$

According to Boundary Conditions

- Initial value: values of the dependent variable or its derivative are known at an initial value
- Boundary value: dependent variable and or derivative are known at more than one point or value of the independent variable

Differential Equations
- **Classification**
- Exercises

ODE
- Analytical Solutions
  - Characteristic Polynomial

According to Boundary Conditions Continued

- Examples:

$$5x^2 \frac{dy}{dx} = 0$$

At x = 0, y = 1 (initial value)

$$5x^2 \frac{d^2y}{dx^2} = 0$$

At x = 0, y = 1

At x = 1 dy/dx =1 (boundary value)

Differential Equations
- **Classification**
- Exercises

ODE
- Analytical Solutions
  - Characteristic Polynomial

More on Classification of Differential Equations

- It is important to know how to classify differential equations to select method of solution
- In the modeling procedure, when you identify mathematical form you must classify your differential equation
- Be prepared to identify and classify any type of differential equation
- The first step to classify differential equations is to identify the dependent and independent variables

Exercises 1 to 4

- Classify the following differential equations according to: type of variable, linearity, homogeneity, and order

$$\frac{d^n x}{dy^n} = y^2 + 1 \quad (1) \qquad \left(\frac{d^2 r}{dy^2}\right)^2 + \frac{d^2 r}{dy^2} + y\frac{dr}{dy} = 0 \quad (2)$$

$$\frac{d^7 b}{dp^7} = 3p \quad (3) \qquad \left(\frac{db}{dp}\right)^7 = 3p \quad (4)$$

Characteristic Polynomial Method

Differential Equations
- Classification
- Exercises

ODE
- Analytical Solutions
  - **Characteristic Polynomial**

- It is used for "linear", homogenous, second order, ODE with constant coefficients (that is, the coefficients do not depend on the independent variable)

Solution Procedure

Differential Equations
- Classification
- Exercises

ODE
- Analytical Solutions
  - **Characteristic Polynomial**

- For the general equation:

$$ay'' + by' + cy = 0$$

1. Define the characteristic polynomial (P) $P = am^2 + bm + c = 0$
2. Calculate the roots of the polynomial
$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
3. Define solution type depending of roots
4. Evaluate the constants using BC's

Solution Type

Differential Equations
- Classification
- Exercises

ODE
- Analytical Solutions
  - **Characteristic Polynomial**

- <u>Case 1</u>: m_1 and m_2 are real and distinct:

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

- <u>Case 2</u>: m_1 and m_2 are real and distinct but symmetric m_1=-m_2:

$$y = A\cosh(m_1 x) + B\sinh(m_1 x)$$

Solution Type Continued

- Case 3: m_1 and m_2 are real and equal (m_1=m_2):

$$y = Ae^{m_1 x} + Bxe^{m_1 x}$$

- Case 4: m_1 and m_2 are complex numbers, and the roots are conjugate pairs (e.g., m_1= p+iq, m_2 = p-iq)

$$y = e^{px}\left[A\cos(qx) + B\sin(qx)\right]$$

Exercise 5

- Find the solution of:

$$y'' - y' - 2y = 0$$
$$\text{at } x=0,\ y=1$$
$$\text{at } x=2,\ y=0$$

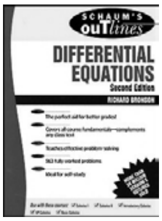
Exercise 6

- Find the solution of:

$$y'' + 4y = 0$$
$$\text{at } x=0,\ y=1$$
$$\text{at } x=\frac{\pi}{4},\ y=0$$

Recommended Reading and Assignment

- Read chapters 1, 2 and 8 (and solve the problems given in those chapters) of:
-



	QA372 .B856 1994eb
	ELECTRONIC BOOK
Author	Bronson, Richard
Title	**Schaum's outline of theory and problems of differential equations [computer file] / Richard Bronson**
Edition	2nd ed
Imprint	New York : McGraw-Hill, c1994

Summary

- Do all the exercises done in class by yourself
- Do recommended assignment
- You must be able to classify differential equations
- You must be able to solve second order, homogenous, nonlinear ODEs